Optics Communications 368 (2016) 190–196

Contents lists available at ScienceDirect

Optics Communications

journal homepage: www.elsevier.com/locate/optcom

Non-Hermitian Hamiltonian and Lamb shift in circular dielectric microcavity

Kyu-Won Park [a,b], Jaewan Kim [b], Kabgyun Jeong [b,c,*]

[a] Department of Physics, Sogang University, Seoul 04107, South Korea
[b] School of Computational Sciences, Korea Institute for Advanced Study, Seoul 02455, South Korea
[c] Center for Macroscopic Quantum Control, Department of Physics and Astronomy, Seoul National University, Seoul 08826, South Korea

ARTICLE INFO

Article history:
Received 15 December 2015
Received in revised form
30 January 2016
Accepted 1 February 2016
Available online 8 February 2016

Keywords:
Non-Hermitian Hamiltonian
Dielectric microcavity
Lamb shift
Whispering gallery mode
Effective potential well

ABSTRACT

We study the normal modes and quasi-normal modes (QNMs) in circular dielectric microcavities through non-Hermitian Hamiltonian, which come from the modifications due to system–environment coupling. Differences between the two types of modes are studied in detail, including the existence of resonances tails. Numerical calculations of the eigenvalues reveal the Lamb shift in the microcavity due to its interaction with the environment. We also investigate relations between the Lamb shift and quantized angular momentum of the whispering gallery mode as well as the refractive index of the microcavity. For the latter, we make use of the similarity between the Helmholtz equation and the Schrödinger equation, in which the refractive index can be treated as a control parameter of effective potential. Our result can be generalized to other open quantum systems with a potential term.

1. Introduction

The so-called quantum billiard systems are conservative closed systems with Dirichlet boundary condition described by Hermitian Hamiltonian with real eigenvalues [1,2]. Once these systems become open, i.e., coupled to their environment, the situation changes. Generally these systems are described by a non-Hermitian Hamiltonian with complex eigenvalues [3–5]. One formalism for treating open systems was developed by Feshbach in 1958 to deal with nuclear decay [6]. Since then the formalism has been used in many other fields such as atomic physics [7], solid state physics [8], and microwave cavities [9].

The Feshbach projection operator formalism yields non-Hermitian Hamiltonian, describing various kinds of interesting phenomena such as bi-orthogonality [10], phase rigidity [11], avoided resonance crossing [12–14], and exceptional points [15–17]. One prominent example is the Lamb shift. It describes a small energy shift in a quantum system caused by the vacuum fluctuations [18–20]. Initially it was observed for a hydrogen atom, but recently the effect has been studied in photonic crystals [21] and cavity QED systems [22]. Here, we generalize this to dielectric microcavities.

As a microcavity is a very attractive optical source in optoelectronic circuits, their emission patterns and high quality factors have been investigated in detail both experimentally and theoretically [23–26]. Importantly, they provide a good platform [27,14] to study the above-mentioned phenomena in open-quantum systems such as bi-orthogonality, phase rigidity, avoided resonance crossing, and exceptional points. We thus investigate the Lamb shift in these systems by comparing the real eigenvalues of the circular quantum billiards and dielectric microwave cavities. Investigations on the openness effects are essential, as exemplified by the concepts such as quasi-scar [28] and Goos–Hänchen shift [29,30].

This paper is organized as follows. In Section 2 we briefly review the theoretical descriptions of quantum billiards and dielectric cavities. This is followed by a summary of Feshbach projection operator (FPO) formalism in Section 3, which yields non-Hermitian Hamiltonians. The resulting quasi-normal modes are compared to the normal modes of quantum billiards in Section 4. Section 5 presents our main results on Lamb shift in a single-layer whispering gallery mode (Section 5.1), and Lamb shift as a function of the refractive index n (Section 5.2). We summarize our results and conclude in Section 6.

2. Quantum billiard and dielectric microcavity

While quantum billiards are completely closed system, the dielectric microcavity is an open quantum system. The eigenvalues of (time-independent) Hamiltonian of a quantum billiard system and those of dielectric microcavity are solutions of Helmholtz

* Corresponding author at: School of Computational Sciences, Korea Institute for Advanced Study, Seoul 02455, South Korea.
E-mail address: kgjeong6@kias.re.kr (K. Jeong).

http://dx.doi.org/10.1016/j.optcom.2016.02.001

equation with different boundary conditions [31] which can be solved by boundary element methods [32]. In quantum billiards, the wave function ψ is described by the Helmholtz equation

$$(\nabla^2 + k^2)\psi(\mathbf{r}) = 0, \tag{1}$$

where k is the wave number and $\psi(\mathbf{r})$ is a wave function such that $\mathbf{r}$ is a position vector inside the billiard system. Its boundary conditions are given by

$$\psi(\mathbf{r} = R) = 0 \quad \text{(Dirichlet)}$$

$$\partial_n\psi(\mathbf{r} = R) = 0 \quad \text{(Neumann)}$$

where R indicates the boundary of the system, ∂_n is the normal derivative to the boundary. Note that we always fix $R=1$ in this paper. The eigenvalues are always real for quantum billiards defined by the above conditions. For a dielectric microcavity with a refractive index n, k^2 of Eq. (1) must be replaced by n^2k^2. In this work, we set n to be 3.3, which is the refractive index of InGaAsP semiconductor microcavity [25]. Therefore, the TE mode $\psi(\mathbf{r})$ obeys the equation

$$\nabla^2\psi(\mathbf{r}) + k^2\mu(\mathbf{r})\psi(\mathbf{r}) = 0, \tag{2}$$

where $\mu(\mathbf{r}) = 1 + (n^2 - 1)\Theta(R - \mathbf{r})$, n is the refractive index of the cavity, $\Theta(\cdot)$ is the unit step function, and its boundary conditions are given by

$$\psi_{in}(\mathbf{r} = R) = \psi_{out}(\mathbf{r} = R),$$

$$\partial_n\psi_{in}(\mathbf{r} = R) = n^2\partial_n\psi_{out}(\mathbf{r} = R).$$

In this paper, we consider only the TE modes. Especially, the solutions of a circular cavity are Bessel functions, classified by a radial quantum number ℓ and an angular quantum number m.

3. Non-Hermitian Hamiltonian description of open quantum system

Typically, quantum systems are not completely closed, i.e., obey the boundary condition above Eq. (1), but are coupled to their environment. In this case, the total Hilbert space consists of the quantum billiard S and an environment E. The total system obeys the Schrödinger equation

$$\boldsymbol{H}|\mathcal{E}\rangle_{SE} = \mathcal{E}|\mathcal{E}\rangle_{SE}, \tag{3}$$

where $\boldsymbol{H}$ is the total Hamiltonian with real energy eigenvalues, and $\mathcal{E}$ is the total energy of the Hamiltonian with the corresponding eigenstate $|\mathcal{E}\rangle_{SE}$. Note that $|\mathcal{E}\rangle$ is the corresponding eigenstates of the energy $\mathcal{E}$. The quantum billiard has a discrete set of states, while we assume that the environment has a continuous set. We can define the projection operators Π_S and Π_E, with $\Pi_S + \Pi_E = \mathbb{1}_{SE}$ and $\Pi_S\Pi_E = \Pi_E\Pi_S = 0$. Here, Π_S is a projection onto the quantum billiard system whereas Π_E is a projection onto the environment, and $\mathbb{1}_{SE}$ is an identity operator defined on the total space [3,4]. In this case, the total Hamiltonian is generally given by

$$\boldsymbol{H} = H_S + H_E + V_{SE} + V_{ES}, \tag{4}$$

where $H_S = \Pi_S\boldsymbol{H}\Pi_S$ and $H_E = \Pi_E\boldsymbol{H}\Pi_E$ are the Hamiltonian of the quantum billiard system and environment, respectively, and $V_{SE} = \Pi_S\boldsymbol{H}\Pi_E \equiv V$ and $V_{ES} = \Pi_E\boldsymbol{H}\Pi_S \equiv V^\dagger$ are interaction Hamiltonians between the system and the environment, respectively: V_{SE} is the interaction from E to S and V_{ES} vice versa.

By exploiting the Hamiltonian [3–5], we can derive an *effective* non-Hermitian Hamiltonian defined solely on quantum billiard system S:

$$H_{\text{eff}} = H_S + V_{SE}G_E^+V_{ES} \tag{5}$$

$$H_{\text{eff}} = H_S - \frac{1}{2}iVV^\dagger + P\int_a^b d\mathcal{E}'\frac{VV^\dagger}{\mathcal{E} - \mathcal{E}'}. \tag{6}$$

Here, G_E^+ is an energy dependent out-going Green function, $VV^\dagger$ is the system interaction via the environment, and P means the principal value. The integral domain $[a, b]$ is known as energy window determined by each decay channels. The decay channels are well-known in a few cases such as rectangular waveguides [4], but in dielectric microcavities, it is only known that the decay channel corresponds to each resonance modes or quasi-normal modes (QNMs) and they are characterized by far-field patterns [13]. Since there is only a single decay channel with respect to each resonance mode, we do not need a summation of decay channels for the principal value in Eq. (6). Eq. (5) can be transformed to Eq. (6) by Sokhotski–Plemelj theorem [33]. After all, this non-Hermitian Hamiltonian has generally complex eigenvalues instead of real eigenvalues:

$$H_{\text{eff}}|\phi_j\rangle = z_j|\phi_j\rangle, \quad z_j = \mathcal{E}_j - \frac{i}{2}\Gamma_j, \tag{7}$$

where z_j is a complex number with $\mathcal{E}_j$ and Γ_j representing the energy and decay width of j-th eigenvector, respectively. Hence, the quality factor Q, as a measure of energy-storing capability, is defined by $Q = \frac{\mathcal{E}_j}{2\Gamma_j}$. In addition, the eigenvectors $|\phi_j\rangle$'s are generally not orthogonal but bi-orthogonal states satisfying $\langle\phi_i^L|\phi_j^R\rangle = \delta_{ij}$ [3]. Note that L and R denote left and right eigenvectors, respectively.

4. Normal mode versus quasi-normal modes

The normal modes are eigenfunctions of the Hermitian Hamiltonian (H_S) with real eigenvalues and are strictly localized in the interior of the system. On the other hand, the quasi-normal modes can leak out into the environment. They can be written as a sum of two parts [3]:

$$|\Omega_j\rangle = |\phi_j\rangle + G_E^+V_{ES}|\phi_j\rangle \tag{8}$$

$$|\Omega_j\rangle \equiv |\phi_j\rangle + |\omega_j\rangle. \tag{9}$$

The $|\phi_j\rangle$ are eigenstates of H_{eff} with generally complex eigenvalues and are localized inside the system, and $|\omega_j\rangle$ describe the resonance tails that reside entirely in the environment. Its presence can be explained as follows. The interaction term V_{ES} gives connection between the system and the environment so that the interior eigenfunction can leak out of the system, which then propagates through environment by out-going Green function G_E^+. As pointed out in Section 3, the decay channels are characterized by far-field patterns, thus we should consider the decay channel as a resonance tail rather than resonance itself, because it is defined only in subspace E. Therefore, this $|\omega_k\rangle$ plays the role of decay channel in dielectric microcavity.

The above description holds well for microcavities as shown in Fig. 1. We can check the above situation from (a) and (b) in Fig. 1. Fig. 1(a) shows a normal mode of circular billiard quantum system with radial quantum number $\ell = 1$, angular quantum $m=2$, and refractive index $n=3.3$. In this case, we found $\text{Re}(kR) \cong 1.556$. The wave function of the quantum billiard system resides entirely within the boundary. It can be conformed by Fig. 1(a), since the brighter points represent the higher probability. Note that the completely black region outside the cavity reflects this fact.

On the other hand, Fig. 1(b) depicts a QNM with same quantum numbers (ℓ and m) and refractive index (n). We can easily identify