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The weighted gyrator transform with its properties and applications

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ABSTRACT

Gyrator transform is a useful tool for information processing that produces twisted rotations in position-spatial frequency planes of phase space and belongs to the class of linear canonical integral transformations. In this paper, we presented a new twisting type of transform by using a different fractionalization process. Unlike the gyrator transform, the newly defined transform cannot be decomposed into a product of two one-dimensional transforms. Therefore it has a better capability of information-tangling than gyrator transform. The transform can be implemented by a photoelectric hybrid setup. Based on the new transform and a chaotic map, we proposed a watermarking scheme of which the viability and performance are demonstrated by means of numerical simulations.

1. Introduction

In the past decades, many more complicated tools than Fourier transform for signal processing such as wavelet [1,2], fractional Fourier transform (FRFT) [3–5] emerged enlarging the realm of the Fourier Optics. Among those tools, the linear canonical integral transformations (LCTs) [6–10] have been significantly developed which mathematically describe the first order optical system. In the past decades, LCTs are widely studied and used in optical and digital information processing such as FRFT [11–14] and gyrator transform (GT) [15,16]. GT is much more recently developed than FRFT which produces a rotation in the twisted position-spatial frequency planes of phase space. Recently, GT is widely applied to information processing especially in image processing due to its nonseparability [17–20]. However, the nonseparability of GT only exists in the space or frequency domain but not in other Wigner distribution planes [21]. GT can be expressed as a 2D FRFT with special angles [15]. Therefore GT can be decomposed as a product of two 1D FRFTs. In this paper, we proposed a novel transform which produces twisted rotations in the phase space, similar to GT, but not belongs to the class of LCT, different from GT. Based on the periodicity of GT, we constructed the new twisted type of transform called the weighted gyrator transform (WGT). WGT is superior to GT regarding to the nonseparability for the reason that WGT is nonseparable in all Wigner distribution planes. Although WGT shares certain similarities with GT, it does not coincide with GT partly due to its different kernel, except in special cases of the integer transform orders. When used in digital and optical information processing, WGT will demonstrate its applicability for its much simpler expression and easier computation and implementation.

The aim of this paper is to present a new type of twisted-transform based on other fractionalization process than that of GT and to demonstrate its applications to some fields of information processing. Because of its similarity to GT, WGT can be applied to optical and digital information processing such as image encryption, digital watermarking, optical filtering and holographic recording. As an example, it was demonstrated the application of WGT for the watermarking in the paper.

In the latter part of this paper, GT is recalled first in Section 2. The analytical expression and properties of WGT are presented in Section 3. The photoelectric hybrid setup implementing WGT is given in this section as well. In the final section we will give numerical simulations to demonstrate how differently WGT acts on a signal from GT. Besides, a watermarking scheme is proposed based on WGT and a chaotic map. The corresponding numerical simulations have shown the feasibility and validity of the watermarking scheme.

2. Gyrator transform

Gyrator transform belongs to LCTs and produces rotations in position-spatial frequency planes. The definition of GT at rotation angle ϕ of a given function $f(\mathbf{r})$ (complex field amplitude on the input plane) is mathematically written as

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$$g(\mathbf{s}) = G^{\phi}[f(\mathbf{r})](\mathbf{s}) = \iint f(x, y)K_G^{\phi}(x, y, u, v)\mathrm{d}x\,\mathrm{d}y \tag{1}$$

where $\mathbf{r} = (x, y)$, $\mathbf{s} = (u, v)$ denote the input/output coordinate respectively here and further. The kernel of the gyrator transform is

$$K_G^{\phi}(x, y, u, v) = \frac{1}{|\sin\phi|}\exp\left(2\pi i\frac{(xy + uv)\cos\phi - (xv + yu)}{\sin\phi}\right). \tag{2}$$

Particularly, when ϕ is equal to 0 or π, $K_G^{\phi} = \delta(\mathbf{r} \mp \mathbf{s})$ and when ϕ is equal to $\pi/2$ or $3\pi/2$, $K_G^{\phi} = \exp[\pm 2\pi i(xv + yu)]$ respectively. The periodicity of GT with respect to the transform angle is an explicit consequence of $K_G^{\phi}(x, y, u, v)$.

GT can be expressed as FRFT with special angles. Let

$$\begin{cases} x = \frac{1}{\sqrt{2}}(s_1 - t_1),\ y = \frac{1}{\sqrt{2}}(s_1 + t_1), \\ u = \frac{1}{\sqrt{2}}(s_2 - t_2),\ v = \frac{1}{\sqrt{2}}(s_2 + t_2). \end{cases} \tag{3}$$

Substitute (3) into Eq. (2), it is obtained that

$$\begin{aligned} K_G^{\varphi}(x, y, u, v) &= \frac{1}{|\sin\varphi|}\exp\left(i\pi\frac{s_1^2\cos\varphi - 2s_1s_2 + s_2^2\cos\varphi}{\sin\varphi}\right) \\ &\quad\times\exp\left(i\pi\frac{t_1^2\cos(-\varphi) - 2t_1t_2 + t_2^2\cos(-\varphi)}{\sin(-\varphi)}\right) \\ &= K_{FRFT}^{\varphi}(s_1, s_2)\cdot K_{FRFT}^{-\varphi}(t_1, t_2), \end{aligned} \tag{4}$$

where $K_{FRFT}^{\phi}(\eta, \xi) = \sqrt{1 - i\cot\phi}\,\exp[i\pi(\eta^2\cos\phi + \xi^2\cos\phi - 2\eta\xi)/\sin\phi]$ is the kernel of 1D FRFT with transform angle ϕ. According to Eq. (4), GT can be taken as a 2D FRFT and decomposed into a product of two 1D FRFTs.

The eigenvalues of GT at angle ϕ are $\lambda_{m,n}(\phi) = \exp[-i\phi(n - m)]$ corresponding to the eigenfunctions $\varphi_{m,n}(\mathbf{r}) = HG_{m,n}(R^{-\pi/4}(\mathbf{r}))$. Here $HG_{m,n}$ denote Hermite–Gaussian functions and $R^{-\pi/4}$ is the rotation operator at angle $-\pi/4$. It is apparent that $\lambda_{m,n}(\phi + 2\pi) = \lambda_{m,n}(\phi)$.

GT at a large range of rotation angles can be realized optically by an optimized flexible system constructed by three generalized lenses with fixed distance between them. We can control the angle ϕ correspondingly by rotating the cylindrical lenses which form the generalized lenses. Other possibility can be achieved by using spatial light modulator. The more detailed description of the experimental realization of the GT is introduced in [22].

3. Weighted gyrator transform

3.1. The weighted gyrator transform

Notice that GT is a periodic operator with periodicity of 2π. Repeated GT of the same transform angle on a given signal will bring the signal back to its original form. Particularly, when the transform orders are integer multiples of $\pi/2$, there are only four different outcomes of a given signal f which are $f_0 = G^0[f] = f(u, v)$, $f_1 = G^{\pi/2}[f] = F(f)(v, u)$, $f_2 = G^{\pi}[f] = f(-u, -v)$, $f_3 = G^{3\pi/2}[f] = F^{-1}(f)(v, u)$ where ‘F’ denotes 2D Fourier transform and ‘F^{-1}’ denotes the inverse 2D Fourier transform. We name these four functions as the basic functions of WGT.

Define weighted gyrator transform with transform order α as

$$G_W^{\alpha}[f](\mathbf{s}) = G_W^{\alpha}[f](u, v) = \sum_{k=0}^{3} C_k(\alpha)f_k(u, v), \tag{5}$$

where $C_k(\alpha)$, $k = 0, 1, 2, 3$ are the coefficients functions of α satisfying three rules:

(1) Continuity condition: $G_W^{\alpha}: L^2(R^2) \to L^2(R^2)$ is continuous;
(2) Boundary condition: G_W^{α} reduces to G^{α} when α is any integer;
(3) Additivity condition: $G_W^{\alpha}G_W^{\beta} = G_W^{\beta}G_W^{\alpha} = G_W^{\alpha+\beta}$ for any real number α and β.

Note that in order to simplify the description, we introduce the term “transform order α” relating the transform angle as $\phi = \pi\alpha/2$. The preceding three rules and the formalism of G_W^{α} imply that

$$C_k(\alpha + \beta) = \sum_{\substack{0\le m,n\le 3 \\ k=\mathrm{mod}(m+n,4)}} C_m(\alpha)C_n(\beta). \tag{6}$$

The boundary condition assigns certain values to the coefficients when α are integers, that is, $C_k(4m + l) = \delta(k - l)$, where m is any integer, $k, l = 0, 1, 2, 3$. According to the continuity condition, the solutions of the equations above are obtained as

$$\begin{aligned} C_k(\alpha) &= \frac{1}{4}\sum_{l=0}^{3}\exp\frac{-i\pi l(\alpha - k)}{2} \\ &= \frac{1}{4}\frac{1 - \exp[-2\pi i(\alpha - k)]}{1 - \exp\frac{-i\pi(\alpha - k)}{2}},\ k = 0, 1, 2, 3. \end{aligned} \tag{7}$$

The coefficients have properties as follows:

(1) $C_k(\alpha) = C_{\mathrm{mod}(k+3,4)}(\alpha - 1)$, $k = 0, 1, 2, 3$;
(2) $\sum_{k=0}^{3} C_k(\alpha) = 1$;
(3) $C_k(\alpha + 4) = C_k(\alpha)$.

Therefore, the expression of WGT at order α is written as

$$G_W^{\alpha}[f](\mathbf{s}) = \sum_{k=0}^{3}\frac{1}{4}\frac{1 - \exp[-2\pi i(\alpha - k)]}{1 - \exp\frac{-i\pi(\alpha - k)}{2}}f_k(u, v). \tag{8}$$

The integral form of WGT on a given signal $f(\mathbf{r})$ is expressed as

$$G_W^{\alpha}[f](\mathbf{s}) = \int K_W^{\alpha}(\mathbf{r}, \mathbf{s})f(\mathbf{r})dr = \int K_W^{\alpha}(x, y, u, v)f(x, y)\mathrm{d}x\,\mathrm{d}y \tag{9}$$

where the kernel $K_W^{\alpha}(x, y, u, v)$ is

$$\begin{aligned} K_W^{\alpha}(x, y, u, v) &= C_0(\alpha)\delta(x - u, y - v) + C_1(\alpha)\exp[-2\pi i(xv + yu)] \\ &\quad + C_2(\alpha)\delta(x + u, y + v) + C_3(\alpha)\exp[2\pi i(xv + yu)]. \end{aligned} \tag{10}$$

WGT presents the spatial information as well as the frequency information of a given signal simultaneously because it is the combination of four basic functions dominating in space, transposed frequency plane and the symmetrical space and transposed frequency plane respectively. Therefore, WGT cannot be decomposed into a product of two 1D FRFTs. Consequently, WGT adequately mixes the spatial and frequency information up due to the presence of the transposed Fourier transform $F[f](v, u)$ and the inverse transposed Fourier transform $F^{-1}[f](v, u)$ in Eq. (8).

3.2. Properties of WGT

WGT satisfies the linearity, additivity, periodicity and unitarity:

(1) Linearity: $G_W^{\alpha}[af(\mathbf{r}) + bg(\mathbf{r})] = aG_W^{\alpha}[f(\mathbf{r})] + bG_W^{\alpha}[g(\mathbf{r})]$, are real numbers;
(2) Additivity: $G_W^{\alpha}G_W^{\beta} = G_W^{\beta}G_W^{\alpha} = G_W^{\alpha+\beta}$;
(3) Periodicity: $G_W^{\alpha+4} = G_W^{\alpha}$;
(4) Unitarity: $G_W^{\alpha}\cdot(G_W^{\alpha})^* = I$, where “*” denotes the complex conjugate transpose operation; a, b
(5) Parseval relation: $\int\{(G_W^{\alpha}f)(\mathbf{s})\}\{(G_W^{\alpha}g)(\mathbf{s})\}^* d\mathbf{s} = \int f(\mathbf{r})g(\mathbf{r})^* d\mathbf{r}$;

The periodicity of WGT is a direct result of periodicity of Fourier transform and the additivity can be concluded from the Eq. (6). Define the coefficients matrix $\mathbf{C}$ of WGT as