Optics Communications 359 (2016) 61–65

Contents lists available at ScienceDirect

Optics Communications

journal homepage: www.elsevier.com/locate/optcom

# Quantum light propagation in longitudinally inhomogeneous waveguides as a spatial Lewis–Ermakov physical invariance

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ARTICLE INFO

*Article history:*
Received 2 July 2015
Accepted 15 September 2015

*Keywords:*
Quantum optics
Wave propagation
Integrated optics

*PACS:*
42.50.-p
42.25.Bs
42.82.-m

ABSTRACT

We study the propagation of quantum states of light in separable longitudinally inhomogeneous waveguides. By means of the usual quantization approach this kind of media would lead to the unphysical result of quantum noise squeezing. This problem is solved by means of generalized canonical transformations in a comoving frame. Under these transformations the generator of propagation is a Lewis–Ermakov invariant in space which is quantized and, accordingly, a propagator consistent with experiments is obtained. Finally, we show that the net effect produced by propagation in these media is a quantum Gouy's phase with applications in quantum information processing and sensing.



## 1. Introduction

Integrated quantum photonics is an active and prolific area of research playing a central role in the development of quantum science due to its scalability and sub-wavelength stability [1]. Major breakthroughs have been recently accomplished in this field, like quantum interference in a directional coupler and the operation of a CNOT gate on-chip [2], integrated quantum metrology with two- and four-photon entangled states [3], reconfigurable photonic quantum chips for processing and measurement of qubits [4,5], quantum walks and boson sampling in waveguides [6,7], quantum teleportation on a photonic circuit [8], continuous-variable entanglement on a quantum circuit [9] and so on. The continuous advance of this technology depends on the use of a correct and consistent theory to design the waveguiding elements which make up the photonic circuits. Quantum propagation problems have been dealt with in homogeneous media, as shown in [10] and references therein. These studies provided the background of the quantum theory of light propagation showing that the operator which describes correctly the quantum spatial propagation along an arbitrary direction $z$ is the Momentum operator $\hat{\mathcal{M}}$, since the Hamiltonian approach fails for problems like dispersive media, counterpropagation and longitudinally inhomogeneous (*LI*) media as well [11]. Likewise, this theory has also been applied to integrated linear and nonlinear coupling devices showing consistent results [11,12].

However, as far as we know, the problem of propagation of quantum light in separable longitudinally inhomogeneous waveguiding media has not been taken into account. These media are widely used in fiber and integrated optics such as phase shifters, modal converters, gradual transitions for anti-reflection, and so on [13–15]; hence the importance of a quantum theory which describes these devices. The analogous problem in the time domain is the time dependent quantum harmonic oscillator. This problem has been thoroughly studied both from mechanical [16,17] and electromagnetic [18,19] points of view, showing squeezing effects. The approach most used in the tackle of this sort of problems is the use of quantum mechanical invariant operators, in particular the Lewis–Ermakov invariant [20]. On the other hand, in the space domain, a first approximation to *LI* media was carried out by Abram [21] and Glauber and Lewenstein [22], where single optical discontinuities between homogeneous media were analyzed. In these studies they showed from energy conservation the different physical behavior the fields experience from the analogous mathematical problem of the time dependent quantum harmonic oscillator, proving that the field quadrature noise does not exhibit real squeezing, in agreement with experiments.

Therefore, our purpose is to give a consistent approach to the propagation in these devices where virtual squeezing is avoided. To this end, we start studying the classical problem which leads to a $z$-dependent Momentum. Since the usual quantization approach

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http://dx.doi.org/10.1016/j.optcom.2015.09.052
0030-4018/

leads to squeezing as in the case of time-dependent problems, but in this case we know from experiments that it is virtual, we look for a reference frame which continuously performs a variable change in the *LI* media and eliminates this virtual squeezing, turning out into a proper propagation generator, a quantum Lewis–Ermakov-type Momentum operator $\hat{\mathcal{M}}$. From this operator we will obtain Fock states in the optical-field strength (OFS) representation $\mathcal{E}$ and derive the propagator, proving the lack of squeezing in this representation and the arising of a quantum Gouy's phase. Furthermore, we will present the particular case of propagation of a Gaussian quantum state in a waveguide with a cosine-type *LI* refractive index, where will be shown that the net effect of this kind of media on quantum states is the generation of a quantum Guoy's phase dependent on features of both the media and the input quantum state, with applications in quantum technology.

## 2. Classical analysis of propagation in longitudinally inhomogeneous waveguides

Our aim is to study the propagation of waveguided modes of quantum light in dispersion-free and non-magnetic media with separable inhomogeneous refractive index in an arbitrary direction of propagation $z$, given by [13]

$$n^2(x, y, z) = n_0^2 f^2(x, y) + \Delta n^2 h^2(z), \tag{1}$$

where the longitudinal $h(z)$ and transversal $f(x, y)$ parts of the index are completely independent and $n_0$ and $\Delta n$ are constants. We focus on the separable index problem as it does not show coupling and therefore radiation modes (losses). From Maxwell equations, it is easy to show that the transverse components of the electric field $\boldsymbol{E}_t(x, y, z, t)$ obey the vectorial wave equation [23]:

$$\nabla^2 \boldsymbol{E}_t + \nabla_t(\boldsymbol{E}\, \nabla(\ln n^2)) = \frac{n^2}{c^2}\frac{\partial^2 \boldsymbol{E}_t}{\partial t^2}. \tag{2}$$

Let us consider monochromatic guided $1D$ vector modes with frequency $\omega_\sigma$ represented by vector field solutions with the following factorable complex amplitudes:

$$\boldsymbol{E}_t(x, y, z, t) = \sum_\sigma q_{\sigma c}(z)\boldsymbol{\xi}_{t\sigma}(x, y)e^{-i\omega_\sigma t}, \tag{3}$$

where we have used $\sigma$ for simplicity standing for the modal numbers $\nu$, $\mu$ in each transverse direction, $z$-dependent complex coefficients $q_{\sigma c}(z)$ fulfilling $\sum_\sigma |q_{\sigma c}(z)|^2 = 1$, and electric normalized transverse complex amplitudes $\boldsymbol{\xi}_{t\sigma}(x, y)$ corresponding to quasi-TE (quasi-TM) modes [11], which belong to the homogeneous part of the refractive index and satisfy

$$\nabla_t^2\, \boldsymbol{\xi}_{t\sigma} + k_0^2 n_0^2 f^2(x, y)\, \boldsymbol{\xi}_{t\sigma} + \nabla_t(\boldsymbol{\xi}_{t\sigma}\, \nabla_t(\ln n_0^2 f^2(x, y))) = \beta_{t\sigma}^2\, \boldsymbol{\xi}_{t\sigma}, \tag{4}$$

with $\beta_t$ being the transverse propagation constant. Solutions of this equation give us the invariant transverse modal structure of the field. Applying Eqs. (1), (3) and (4) into (2), we obtain

$$\frac{d^2 q_\sigma}{dz^2} + \beta_\sigma^2(z) q_\sigma = 0, \tag{5}$$

where $q_\sigma = (q_{\sigma c} + q_{\sigma c}^*)/2$ stands for the real electric field coefficients, $\beta_\sigma^2(z) = \beta_{t\sigma}^2 + k_0^2 \Delta n^2 h^2(z)$ is the local propagation constant of the $\sigma$-mode and where we have used the approximation $E_z \ll E_x, E_y$. It is important to outline that $E_z = 0$ in the case of TE modes, that is, Eq. (5) is exact for such modes. This propagation equation clearly suggests a local spatial harmonic oscillator and therefore it can be directly derived from spatial-type Hamilton equations where the Hamiltonian is substituted by the Momentum, since it is the generator of spatial translations [12,21], given by

$$\mathcal{M}_\sigma = \frac{1}{2}[p_\sigma^2 + \beta_\sigma^2(z) q_\sigma^2], \tag{6}$$

with $p_\sigma = q'_\sigma$ and prime stands for $z$-derivative. This result is analogous to that obtained in [18] where time-dependent linear media was studied. The classical Momentum (6) is equivalent to the Hamiltonian of a time-dependent harmonic oscillator, with $\beta(z)$ playing the role of $\omega(t)$ [20].

Likewise, the solution of Eq. (5) is easily obtained via the use of the complex electric field $q_{\sigma c}$ in the following way:

$$q_{\sigma c}(z) = \rho_\sigma\, e^{i\theta_\sigma} q_{\sigma c}(0), \tag{7}$$

where $\rho_\sigma$ and $\theta_\sigma$ are real functions obtained by solving:

$$\frac{d^2 \rho_\sigma}{dz^2} + \beta_\sigma^2(z) \rho_\sigma = \frac{\beta_{0\sigma}}{\rho_\sigma^3}, \tag{8}$$

$$\frac{d\theta_\sigma}{dz} = \frac{\beta_{0\sigma}}{\rho_\sigma^2}, \tag{9}$$

with $\beta_{0\sigma} \equiv \beta_\sigma(0)$. Eq. (8) is an Ermakov–Pinney equation with solutions given by [24]

$$\rho_\sigma(z) = [(\beta_{0\sigma}\, u_\sigma(z))^2 + v_\sigma^2(z)]^{1/2}, \tag{10}$$

$$\rho_\sigma(0) = 1, \quad \rho'_\sigma(0) = 0, \tag{11}$$

and where $u_\sigma$ and $v_\sigma$ are linearly independent functions that satisfy Eq. (5) and have the following initial conditions and Wronskian:

$$u_\sigma(0) = v'_\sigma(0) = 0, \tag{12}$$

$$u'_\sigma(0) = v_\sigma(0) = 1, \tag{13}$$

$$W_\sigma = u'_\sigma\, v_\sigma - v'_\sigma u_\sigma = 1. \tag{14}$$

So, the classical wave amplitude changes continuously during propagation by a factor $\rho_\sigma$, whereas the modal propagation constant is given by $\tilde{\beta}_\sigma \equiv \theta'_\sigma$ via Eq. (9) where $\theta_\sigma$ represents the total phase accumulated in the propagation.

## 3. Quantization in longitudinally inhomogeneous waveguides

The next step is to quantize the classical Momentum (6). But before carrying out this step, some considerations have to be taken into account. Direct quantization of the classical fields $q_\sigma$ and $p_\sigma$ would lead to a quantized $z$-dependent Momentum analogous to the Hamiltonian for a time-dependent harmonic oscillator and, therefore, following the usual steps, we would have propagation equations leading to quadrature noise squeezing [16–19]. But in the study of propagation in *LI* dielectric media, this approach does not provide consistent results. As was pointed out by Abram in his seminal paper about quantization of light in dielectric media [21], when quantum states propagating in a dielectric are represented in the basis of free-space photons, they seem to be squeezed, but they are produced by energy non-conserving terms and then inside a dielectric there is no experiment that can detect these photons. This happens because if we wish to detect photons inside the medium the fields would experiment an effective refractive index equivalent to the squeezing parameter and, therefore, a scale change is necessary. In the same spirit an interesting discussion about quantization in a dielectric was carried out by Glauber and Lewestein in [22]. In this study it is stressed that the measurement