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Superoscillatory field features with evanescent waves

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ABSTRACT

We show how to obtain optical fields possessing superoscillatory features by superposing the evanescent tails of waves undergoing total internal reflection at a plane dielectric interface. In doing so, we essentially extend the definition of superoscillations to functions expressed as a continuum of slowly decaying exponentials, while not necessarily being bandlimited in the standard (Fourier) sense. We obtain such functions by complexifying the argument of standard bandlimited superoscillatory functions with a strictly positive spectrum. Combined with our recent method for superoscillations with arbitrary polynomial shape, the present approach offers flexibility for locally shaping the evanescent field near dielectric interfaces for applications such as particle or atom trapping.

The phenomenon of superoscillations implies the ability of bandlimited signals to oscillate with local frequencies that are arbitrarily larger than their maximum frequency component. This counter-intuitive property has long intrigued physicists and engineers because it enables, at least theoretically, antenna or imaging systems to produce or resolve wave features much finer than their bandwidth suggests. The earliest reports of the phenomenon can therefore be found in classical works on superdirectivity [1] and subdiffraction imaging [2], as well as in information and signal theory [3,4]. The underlying idea of all these works was that extremely fine (or fast) oscillations can be obtained from appropriately weighted superpositions of slowly-varying functions whose rate of variation is limited by some fundamental bandwidth cutoff. The latter is introduced either by the operating wavelength in radiation or imaging settings or by the response time of circuits and systems in signal processing settings. Interestingly, the term *superoscillations* itself was coined independently in quantum mechanics in the context of weak measurements [5] and was thereafter established when referring to such *faster-than-Fourier* functions [6]. The interest in superoscillations and particular in superoscillatory optical imaging has recently revived with a number of promising experimental demonstrations [7,8].

Superoscillations have so far been obtained as superpositions of *propagating* waves in homogeneous media. From a theoretical viewpoint, this is due to their original definition within a space of bandlimited functions, namely functions whose Fourier transform has an appropriate compact support. In this context, a superoscillatory function $f(x)$ is always expressible (through its inverse Fourier transform) as a superposition of propagating waves e^{ikx} with k being limited within a given bandwidth. From the applications viewpoint too, creating superoscillations with propagating waves has been the major focus of research with the aim of achieving optical superresolution without evanescent waves or, equivalently, subwavelength focusing of light in the far field [9,10].

In this communication we consider a different question: Is it possible to produce superoscillatory field features using as basis functions nonpropagating evanescent waves of the form e^{-kx}? Before we give an answer, the concept of superoscillations within the context of evanescent waves should first be clarified (or better, introduced). In standard superoscillations, one considers functions expressed as superpositions of propagating waves e^{ikx} over a finite range of k (bandlimited functions) which can oscillate at spatial scales s with $k_{max} s$ âª¡ 1, with k_{max} being the maximum frequency in the spectrum. We use a similar concept for evanescent waves e^{-kx}. These have also a clearly defined spatial scale k^{-1}, hence if a superposition of such waves could oscillate locally at a scale s âª¡ k_{max}^{-1}, e.g. like $\sin(\pi x/s)$, this would also qualify as a superoscillation, in analogy to the case of propagating waves e^{ikx}. In other words, we consider the possibility of obtaining fast oscillations (in particular, densely spaced zeros) by superposing slowly decaying exponentials instead of slowly oscillating sines and cosines. This possibility can be thought of as extending the definition

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of superoscillations to functions that are amenable to a Laplace but not necessarily to a Fourier transform (and thus may not be bandlimited in the strict sense), which may be an interesting extension both from the physics and the mathematics point of view.

Specifically consider a function defined in $x \geq 0$ through a continuous superposition of evanescent waves with decay constants k limited in some range $[k_1, k_2]$ as

$$f(x) = \int_{k_1}^{k_2} F(k) e^{-kx}\, dk, \quad x \geq 0 \tag{1}$$

where $F(k)$ is a (generally complex) amplitude function. This function remains undefined in $x < 0$ hence one cannot readily speak of its Fourier transform or its bandwidth. Of course, one can assume that $f(x)$ is extended to $x < 0$ through its analytic continuation (simply by using Eq. (1) with negative x) in which case the function becomes unbounded as $x \to -\infty$ and thus non-Fourier transformable. Other choices lead to a Fourier transformable function, such as an even or odd reflection ($f(-x) = f(x)$ or $f(-x) = -f(x)$) or the causal case $f(x) = 0$ for $x < 0$. Whatever the choice, the values of $f(x)$ in $x < 0$ are immaterial to our problem which is to obtain fast oscillations from a superposition of slowly decaying exponentials. In a physical setting, as for example when $f(x)$ is the wave along the x-axis obtained after total internal reflection (TIR) at an interface at $x=0$, this implies that we do not pose any restrictions on the field or the medium on the high-index side ($x < 0$) other than it must produce the desired $f(x)$ in $x > 0$.

Hence the question is whether a function of the type defined in Eq. (1) can actually oscillate at small spatial scales, namely superoscillate. The answer is obtained by associating $f(x)$ with the analytic continuation of a bandlimited function $\Phi(\xi)$ (of the real variable ξ) whose Fourier transform is supported on $[k_1, k_2]$ and equals $F(k)$. Such a function is expressed as

$$\Phi(\xi) = \int_{k_1}^{k_2} F(k) e^{ik\xi}\, dk, \quad \xi \text{ real} \tag{2}$$

According to the Paley–Wiener theorem [11], the complex function $\Phi(z)$, that is obtained from Eq. (2) by complexifying the argument ξ to $z = \xi + i\eta$, is an entire function (of exponential type, namely there exists $C > 0$ such that $|g(z)| \leq Ce^{k_2|z|}$). Therefore, we can obtain from Eq. (2) analytic functions $f(x)$ of a real variable x by letting z run along any smooth curve $z(x)$ on the complex plane. Notice that, for a general path $z(x)$, these functions will be free of singularities but not necessarily square integrable (square integrability is guaranteed by the Paley–Wiener theorem only for paths parallel to the real axis [11]). In particular, if we restrict z to the imaginary axis ($z = ix$), a propagating wave $e^{ik\xi}$ turns into an evanescent wave e^{-kx} and Eq. (2) gives the $f(x)$ of Eq. (1) in $x \geq 0$, namely

$$f(x) = \Phi(ix), \quad x \geq 0 \tag{3}$$

We have therefore reached an important conclusion. A $f(x)$ defined through Eq. (1) can be viewed as the restriction to the positive imaginary axis of an entire function $\Phi(z)$, whose restriction to the real axis $\Phi(\xi)$ has the Fourier transform $F(k)$ supported on $[k_1, k_2]$. And by the uniqueness of an analytic continuation, the complex analytic function $\Phi(z)$ satisfying Eq. (3) is unique. In this way, the oscillations of $f(x)$ are mapped to oscillations of $\Phi(z)$ along the positive imaginary axis. These oscillations can in turn be controlled by the zeros of $\Phi(z)$. And here is where a remarkable property of entire functions applies. The zeros of entire functions $\Phi(z)$, which are defined as the extension in the complex plane of a bandlimited function $\Phi(\xi)$, can be placed arbitrarily close to each other no matter how small is the maximum frequency contained in $\Phi(\xi)$ [12]. In our case of $\Phi(\xi)$ defined as in Eq. (2), this maximum frequency is k_2. This property of entire functions has also been used in the past to produce standard superoscillatory functions [13,14]. One can therefore construct $\Phi(z)$ to have a set of densely spaced zeros along the positive imaginary axis, $z_n = ix_n$, $x_n \geq 0$, $n = 1, 2, \ldots$, to obtain a function $f(x) = \Phi(ix)$ having densely spaced zeros at $x = x_n$. We have therefore shown that $f(x)$ can indeed superoscillate in the aforementioned sense.

Let us illustrate the above with an example. Consider the entire function

$$\Phi(z) = \frac{(z - is)(z - 2is)(z - 3is)}{(z - 1)(z - 2)(z - 3)} \text{sinc}(z) e^{i\delta z} \tag{4}$$

where s âª¡ 1, $\delta > \pi$ and $\text{sinc}(z) = \sin(\pi z)/(\pi z)$. The logic behind this $\Phi(z)$ is, firstly, to replace the three zeros of the entire function $\text{sinc}(z)$ at $z = 1, 2, 3$ by three closely spaced zeros along the imaginary axis at $z = is, 2is, 3is$. The new function is also entire and of the same exponential order $e^{\pi|z|}$ with $\text{sinc}(z)$ hence, according to the zero-replacement theorem [14], its restriction to the real axis has the same bandwidth (2π) with $\text{sinc}(\xi)$. Secondly, the exponential factor $e^{i\delta z}$ of $\Phi(z)$ becomes on the real axis the phase factor $e^{i\delta\xi}$ which serves to shift the spectrum of $\Phi(\xi)$ from $[-\pi, \pi]$ to the strictly positive wavenumbers $[\delta - \pi, \delta + \pi]$. Now, according to our previous discussion, the restriction of $\Phi(z)$ to the positive imaginary axis, i.e.,

$$f(x) = \Phi(ix) = \frac{(x - s)(x - 2s)(x - 3s)}{(x + i)(x + 2i)(x + 3i)} \frac{\sinh(\pi x)}{\pi x} e^{-\delta x}, \quad x \geq 0 \tag{5}$$

is a function that is expressible as in Eq. (1) with $k_1 = \delta - \pi$ and $k_2 = \delta + \pi$ and with $F(k)$ being the Fourier transform of $\Phi(\xi)$. Moreover $f(x)$ superoscillates at the (arbitrarily small) scale s due to the three zeros at $x = s, 2s, 3s$. Note also the decay of $f(x) \sim e^{(\pi-\delta)x}/x$ as $x \to +\infty$. An example of this $f(x)$ is shown in Fig. 1(a) for $s=0.1$ and $\delta=1.5\pi$. Here s has been deliberately chosen large for illustration purposes. The corresponding complex amplitude function $F(k)$ follows by numerically computing the Fourier transform of the $\Phi(\xi)$ of Eq. (4) (using the inverse to Eq. (2) relation) and is shown in Fig. 1(b).

The design of superoscillations using evanescent waves has therefore been reduced to the design of standard bandlimited superoscillatory functions with a strictly positive spectrum. There are two general methods available in the literature for constructing standard superoscillatory functions, apart from the zero-replacement theorem that we have just used. The first method expresses the function as a finite sum of sinc functions with coefficients that are determined by imposing a set of amplitude or derivative constraints over a finite grid of closely spaced points [15–17]. The second method uses similar constraints but the superoscillatory function is expressed as a Fourier integral and a minimum-energy solution is sought using variational techniques [18,19]. These methods have a *discrete* logic in the sense that the superoscillatory function is constrained over a discrete set of points. This however implies some lack of control over the actual shape of the superoscillatory curve across the interval that contains these points (superoscillatory interval).

Towards a continuous design of superoscillations, we have recently proposed a simple method for creating superoscillations that mimic a given polynomial with arbitrarily high precision within some finite interval [20]. Such superoscillatory functions are obtained as the product of the target polynomial with a bandlimited envelope function whose Fourier transform has at least $N-1$ continuous derivatives and a Nth derivative of bounded variation, with N being the order of the polynomial. For example consider the entire function

$$\Phi(z) = -(iz + s)(iz + 2s)(iz + 3s)\,\text{sinc}^4\left(\frac{z}{4}\right) e^{i\delta z} \tag{6}$$