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On the stress singularities generated by anisotropic eigenstrains and the hydrostatic stress due to annular inhomogeneities

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ABSTRACT

The problems of singularity formation and hydrostatic stress created by an inhomogeneity with eigenstrain in an incompressible isotropic hyperelastic material are considered. For both a spherical ball and a cylindrical bar with a radially symmetric distribution of finite possibly anisotropic eigenstrains, we show that the anisotropy of these eigenstrains at the center (the center of the sphere or the axis of the cylinder) controls the stress singularity. If they are equal at the center no stress singularity develops but if they are not equal then stress always develops a logarithmic singularity. In both cases, the energy density and strains are everywhere finite. As a related problem, we consider annular inclusions for which the eigenstrains vanish in a core around the center. We show that even for an anisotropic distribution of eigenstrains, the stress inside the core is always hydrostatic. We show how these general results are connected to recent claims on similar problems in the limit of small eigenstrains.

1. Introduction

The general problem of elastic inclusion (or more generally, inhomogeneity with eigenstrain) is to compute the stress generated by adding material within a given matrix. Mathematically, it can be formulated as a problem where eigenstrains, which represent the new included material, are given and for which the residual stress needs to be computed (see Yavari and Goriely, 2013 and references therein for a general introduction on the topic of inclusions, eigenstrains, and various extensions of the celebrated work of Eshelby (1957)).

In general, the eigenstrains do not need to be isotropic with respect to the symmetry of the underlying system. For instance in a ball, the spherical solution still exists even if different eigenstrains in the radial and angular directions are specified. More specifically, in Yavari and Goriely (2013), we analyzed a ball of radius R_o with a spherical inclusion of radius R_i with uniform radial and circumferential (finite) eigenstrains. The matrix and the inclusion both were assumed to be incompressible and isotropic with possibly different energy functions. It was observed that when the uniform radial and circumferential eigenstrains are not equal, i.e. an anisotropic eigenstrain distribution, the non-vanishing stress components all have a logarithmic singularity at the center of the ball $R=0$. However, the principal stretches and hence the strain energy density are finite everywhere.

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It has been known for a long time that certain anisotropies in elastic properties can lead to stress singularities even for bodies with smooth boundaries. The first such observations were made by Lekhnitskii (1957) and Reissner (1958). Lekhnitskii (1957) showed that the stress on the axis of a cylindrically uniform solid cylinder made of a monoclinic solid may become infinite under a finite uniform applied pressure. Here, cylindrically uniform means that the elastic constants in cylindrical coordinates are constant. Reissner (1958) observed similar singularities in the case of orthotropic shells of revolution. Later, Antman and Negrón-Marrero (1987) studied the radially symmetric equilibrium configurations of transversely isotropic solid cylinders and balls under constant pressure on their boundaries and showed that for applied pressure above a critical value, pressure at the center may become unbounded.

Avery and Herakovich (1986) analyzed a linear elastic cylindrically anisotropic circular cylindrical bar under uniform thermal load. They showed that in the case of radial orthotropy (radial stiffness larger than hoop stiffness) the stress develops a singularity on the axis of the bar. Gal and Dvorkin (1995) considered an anisotropic cylindrical bar with uniform finite tractions on the boundary. They showed that if the cylinder is stiffer radially than tangentially (radially orthotropic cylinder) stress on the axis of the cylinder becomes unbounded. Ting (1953) considered a spherically uniform (i.e. elastic constants in the spherical coordinates are constant) linear anisotropic spherical ball under uniform pressure on its boundary sphere and showed that for certain anisotropies the stress at the center of the ball is unbounded. Later Aguiar (2006) observed that in a neighborhood of the origin the Jacobian is negative in Ting's solution and hence the solution is unphysical.[1] Aguiar (2006) used Fosdick and Royer-Carfagni (2001)'s framework for avoiding self-intersection of matter and observed that the corresponding Lagrange multiplier has a logarithmic singularity at the center of the ball. It seems that in all these examples anisotropy in a neighborhood of the origin (of cylindrical or spherical coordinates) is responsible for stress singularities (see also Horgan and Baxter, 1996). More recently, Goriely et al. (2010) showed that, in morphoelasticity, non-isotropic growth in a ball or cylinder always leads to stress singularity and Sadik and Yavari (2014) showed that anisotropic thermal expansion induces logarithmic singularities as well.

At first sight, a singularity in the stress field may appear unphysical. It could be seen as an artifact of the mathematical model, related to the peculiar choice of coordinates. Although the stress field is an important physical construct, it is only through tractions that forces are exerted on the material. As long as the actual physical forces developed in the material remain finite, that the strain energy is bounded, and that the material does not interpenetrate, a solution with singularity is a valid physical solution for the problem at hand. Further, the setting in which these singularities develop may represent a challenge from a computational point of view. It is therefore particularly important to classify these solutions analytically so that their occurrence in a numerical scheme could be controlled locally.

The present work was motivated by two recent papers. First the paper by Shodja and Khorshidi (2013) where stress singularities are observed in the framework of linear elasticity. The question raised, by Markenscoff and Dundurs (2014), was whether these singularities can exist at all for small strains. To settle the matter, we will compute the exact nonlinear solution and show that indeed, in the limit of small strains, it is consistent with the solution of Shodja and Khorshidi (2013). We will further generalize the problem and identify the origin of stress singularity in cylindrical and spherical geometries. Second, the paper of Markenscoff and Dundurs (2014) who studied annular inhomogeneities with eigenstrains. The authors considered both spherical and cylindrical geometries and assumed that the eigenstrains in the inhomogeneities to be pure dilatational and positive. They showed that when the shear modulus of the annular inhomogeneity is larger than that of the core, tensile hydrostatic stress is created in the core. We revisit this problem by computing the exact solution. We show, among other results, that when the strain-energy density functions of the inhomogeneity and the core (and matrix) are identical, the stress inside the core does not necessarily vanish. However, it vanishes to first order in the eigenstrains where Markenscoff and Dundurs (2014)'s result is recovered. These problems of eigenstrains and singularity formation in elastic materials can be very subtle and their interpretation and validity may be clouded by the approximations made to obtain them. In such exceptional cases where an exact solution can be obtained and various limits explicitly computed, no such doubts persist.

2. Logarithmic stress singularities generated by finite anisotropic eigenstrains in a spherical ball

We first briefly review the problem solved in Yavari and Goriely (2013). Consider a spherical ball of radius R_0 made of an incompressible isotropic body with an energy function that may explicitly depend on R (in the spherical coordinates (R, Θ, Φ)). We assume that there are (finite) eigenstrains in the ball that may induce residual stresses. We assume that the radial and circumferential eigenstrains $e^{\omega_R(R)}$ and $e^{\omega_\Theta(R)}$ are given and that $\omega_R(R)$ and $\omega_\Theta(R)$ are analytic in a neighborhood of the origin. Yavari and Goriely (2013) assumed that the ball is stress free in the absence of eigenstrains for which the flat metric in the material manifold reads $\mathbf{G}_0(\mathbf{X}) = \mathbf{G}_0(R) = \mathrm{diag}(1, R^2, R^2\sin^2\Theta)$. In the presence of eigenstrains the material manifold (where the ball is stress free) has the Riemannian metric $\mathbf{G}(\mathbf{X}) = \mathbf{G}(R) = \mathrm{diag}(e^{2\omega_R(R)}, e^{2\omega_\Theta(R)}R^2, e^{2\omega_\Theta(R)}R^2\sin^2\Theta)$. Using the spherical coordinates (r, θ, ϕ) for the Euclidean ambient space, looking for solutions of the form $(r, \theta, \phi) = (r(R), \Theta, \Phi)$, assuming incompressibility, and $r(0) = 0$, one obtains $r(R) = \left(\int_0^R 3\xi^2 e^{\omega_R(\xi)+2\omega_\Theta(\xi)}\, d\xi\right)^{1/3}$. The principal stretches read $\lambda_1 = (R^2/r^2(R))e^{2\omega_\Theta(R)}$, $\lambda_2 = \lambda_3 = (r(R)/R)e^{-\omega_\Theta(R)}$. For an (inhomogeneous) isotropic solid the strain-energy density function

[1] Note that in the examples that Yavari and Goriely (2013) solved $J=1$ everywhere and hence there is no interpenetration of matter anywhere.