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A numerical basis for strain-gradient plasticity theory: Rate-independent and rate-dependent formulations

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ABSTRACT

A numerical model formulation of the higher order flow theory (rate-independent) by Fleck and Willis [2009. A mathematical basis for strain-gradient plasticity theory – part II: tensorial plastic multiplier. Journal of the Mechanics and Physics of Solids 57, 1045-1057.], that allows for elastic–plastic loading/unloading and the interaction of multiple plastic zones, is proposed. The predicted model response is compared to the corresponding rate-dependent version of visco-plastic origin, and coinciding results are obtained in the limit of small strain-rate sensitivity. First, (i) the evolution of a single plastic zone is analyzed to illustrate the agreement with earlier published results, whereafter examples of (ii) multiple plastic zone interaction, and (iii) elastic–plastic loading/unloading are presented. Here, the simple shear problem of an infinite slab constrained between rigid plates is considered, and the effect of strain gradients, strain hardening and rate sensitivity is brought out. For clarity of results, a 1D model is constructed following a procedure suitable for generalization to 2D and 3D.

1. Introduction

Experimental observations of additional hardening (e.g. Xiang and Vlassak, 2006) and increased yield resistance (e.g. Fleck et al., 1994; Swadener et al., 2002) at the micron scale have led to a vast amount of theoretical literature seeking to model such effects. It has been recognized that strain gradients are the reason for the size effects observed, and a physical explanation is achieved by the concept of Geometrically Necessary Dislocations (GND's), which affects the plastic behavior, in addition to the so-called Statistically Stored Dislocations (SSD). A classical example is curvature resulting from bending in the plastic regime, where GND's offer a simple explanation for the material compatibility. The GND density can be related to the lattice curvature, and it is known to provide macroscopic strengthening (Ashby, 1970; Russel and Ashby, 1970).

In spite of experimental evidence and insight into the mechanisms involved, *it has not been a simple matter to obtain a sound extension to the classical J2 flow theory of plasticity that incorporates a dependence on plastic strain gradients* (stated by Hutchinson, 2012). Nevertheless, a number of phenomenological strain gradient enhanced flow theories involving higher order stresses that are work-conjugate to the strain gradients have been proposed (Fleck and Hutchinson, 1997, 2001; Gudmundson, 2004; Gurtin and Anand, 2005, 2009; Fleck and Willis, 2009; Niordson and Hutchinson, 2011), all suffering from individual drawbacks that arise with the difficulty in accounting for strain gradients in a consistent manner, e.g., the widely used Fleck–Hutchinson theory (2001) does not guarantee positive plastic dissipation for specific straining histories (Gurtin and Anand, 2009), whereas the theory of Gudmundson (2004), Gurtin and Anand (2005) and Fleck and Willis (2009)

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predicts discontinuous changes in the higher order stresses upon certain infinitesimal load changes. It is emphasized that the present work does not intend to clarify these fundamental issues. However, a recent discussion can be found in Hutchinson (2012).

The class of higher order flow theories including Aifantis (1984), Muhlhaus and Aifantis (1991), Fleck and Hutchinson (1997) and Fleck and Hutchinson (2001) has the advantage that the numerical formulation is straight forward, thus being well-suited for numerical solution techniques, such as the finite element method. This advantage has contributed to a wide use. On the other hand, significant challenges are encountered for the rate-independent flow theory by Gudmundson (2004), Gurtin and Anand (2005) and Fleck and Willis (2009). The problems in this class of theories involve the definition of yielding in relation to the numerical implementation, and the treatment of the evolution and interaction of multiple plastic zones. The objective of this work is to present and demonstrate a novel numerical formulation of the higher order flow theory (rate-independent) by Fleck and Willis (2009), that allows for elastic–plastic loading/unloading and the interaction of multiple plastic zones. Here, the aim is threefold: (i) to analyze the evolution of a single plastic zone and ensure agreement with earlier published results, (ii) to validate the predictability in terms of the evolution and interaction of multiple plastic zones, and (iii) to ensure correct treatment of elastic–plastic loading/unloading. Throughout, the model response is compared to the corresponding rate-dependent visco-plastic version of the Fleck–Willis theory, which acts as an ideal basis for comparison, and fortunately lends itself nicely to numerical implementation (Niordson and Legarth, 2010; Niordson and Hutchinson, 2011; Danas et al., 2012; Dahlberg and Faleskog, 2013). To facilitate easy management of multiple plastic zones a layered infinite material slab, constrained between rigid platens and loaded in simple shear, is considered. For clarity of results, a 1D model is constructed using a procedure that leans on the formulation of the corresponding rate-dependent theory combined with classical image analysis, which is suitable for generalization to 2D and 3D. The paper is structured as follows. The theoretical basis of the generalized flow theory (rate-independent) by Fleck and Willis (2009) and its rate-dependent counterpart is outlined in Section 2. Here, focus is on the variational principles to be used in the numerical formulation and the modeling procedure which is presented in Section 3. A 1D shear model is developed in Section 4, and predicted results are given in Section 5. The work is concluded in Section 6.

2. Strain gradient theory: rate-independent and rate-dependent

The present work builds upon a small strain theory for strain gradient plasticity developed by Gudmundson (2004), Gurtin and Anand (2005) and Fleck and Willis (2009). A compact overview of the generalized flow theory (rate-independent) and the corresponding visco-plastic formulation (rate-dependent) is given below. Throughout, Einstein's summation rule is utilized in the tensor equations and $()_{,i}$ denotes partial differentiation with respect to the spatial coordinate x_i.

2.1. Variational principles and constitutive relations

Employing a small strain formulation, the total strain rate is determined from the gradients of the displacement rates; $\dot{\varepsilon}_{ij} = (\dot{u}_{i,j} + \dot{u}_{j,i})/2$, and decomposes into an elastic part, $\dot{\varepsilon}^e_{ij}$, and a plastic part, $\dot{\varepsilon}^p_{ij}$, so that; $\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^p_{ij}$. For a gradient enhanced material, involving higher order stresses, the principle of virtual work reads (Gudmundson, 2004)

$$\int_V (\sigma_{ij}\delta\varepsilon_{ij} + (q_{ij} - s_{ij})\delta\varepsilon^p_{ij} + \tau_{ijk}\delta\varepsilon^p_{ij,k})\,\mathrm{d}V = \int_S (T_i\delta u_i + t_{ij}\delta\varepsilon^p_{ij})\,\mathrm{d}S. \tag{1}$$

Here, σ_{ij} is the symmetric Cauchy stress tensor and $s_{ij} = \sigma_{ij} - \delta_{ij}\sigma_{kk}/3$ its deviatoric part. In addition to conventional stresses, the principle of virtual work incorporates the so-called micro-stress tensor, q_{ij} (work-conjugate to the plastic strain, ε^p_{ij}), and the higher order stress tensor, τ_{ijk} (work-conjugate to plastic strain gradients, $\varepsilon^p_{ij,k}$). The right-hand side of Eq. (1) thereby includes both conventional tractions, $T_i = \sigma_{ij}n_j$, and higher order terms, $t_{ij} = \tau_{ijk}n_k$, with n_k denoting the outward normal to the surface S.

The mechanisms associated with dislocation movement and/or storage of GND's (Ashby, 1970; Gurtin, 2002; Ohno and Okumara, 2007) have been incorporated into the current higher order theory by assuming the micro-stress to have a dissipative part only; $q_{ij} = q^D_{ij}$, while the higher order stresses decompose into a dissipative part, τ^D_{ijk}, and an energetic part, τ^E_{ijk}, such that $\tau_{ijk} = \tau^D_{ijk} + \tau^E_{ijk}$. Thus, an assumption for the free energy can be introduced according to the isotropic expression

$$\Psi = \frac{1}{2}\left(\varepsilon_{ij} - \varepsilon^p_{ij}\right)\mathcal{L}_{ijkl}(\varepsilon_{kl} - \varepsilon^p_{kl}) + \frac{1}{2}G(L_E)^2\varepsilon^p_{ij,k}\varepsilon^p_{ij,k} \tag{2}$$

whereby the conventional stresses is given through the elastic relationship; $\sigma_{ij} = \partial\Psi/\partial\varepsilon^e_{ij} = \mathcal{L}_{ijkl}(\varepsilon_{kl} - \varepsilon^p_{kl})$, while the energetic higher order stresses are $\tau^E_{ijkl} = \partial\Psi/\partial\varepsilon^p_{ij,k} = G(L_E)^2\varepsilon^p_{ij,k}$. Here, $\mathcal{L}_{ijkl}$ is the isotropic elastic stiffness tensor, G is the elastic shear modulus and L_E is an isotropic energetic constitutive length parameter. Although, the energetic length parameter is taken to be zero throughout this study, the numerical framework is developed to handle energetic contributions.

Introducing a quadratic form of the gradient enhanced effective plastic strain rate, given by

$$\dot{E}^p = \sqrt{\frac{2}{3}\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ij} + (L_D)^2\dot{\varepsilon}^p_{ij,k}\dot{\varepsilon}^p_{ij,k}} \tag{3}$$