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An energy formulation of continuum magneto-electro-elasticity with applications

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ABSTRACT

We present an energy formulation of continuum electro-elasticity and magneto-electro-elasticity. Based on the principle of minimum free energy, we propose a form of total free energy of the system in three dimensions, and then systematically derive the theories for a hierarchy of materials including dielectric elastomers, piezoelectric ceramics, ferroelectrics, flexoelectric materials, magnetic elastomers, magnetoelectric materials, piezoelectric–magnetic materials among others. The effects of mechanical, electrical and magnetic boundary devices, external charges, polarizations and magnetization are taken into account in formulating the free energy. The linear and nonlinear boundary value problems governing these materials are explicitly derived as the Euler–Lagrange equations of the principle of minimum free energy. Finally, we illustrate the applications of the formulations by presenting solutions to a few simple problems and give an outlook of potential applications.

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1. Introduction

The classic subjects of electrostatics, magnetostatics and elasticity have been firmly established in the last few centuries. For a continuum deformable body $\Omega \subset \mathbb{R}^3$, the electric, magnetic and elastic state of the body in static equilibrium necessarily satisfy the Maxwell equations and mechanical balance laws:

$$\operatorname{curl} \mathbf{e} = 0, \quad \operatorname{div} \mathbf{d} = \rho^e, \quad \mathbf{d} = \epsilon_0 \mathbf{e} + \mathbf{p} + \mathbf{p}^e, \tag{1.1a}$$

$$\operatorname{curl} \mathbf{h} = 0, \quad \operatorname{div} \mathbf{b} = 0, \quad \mathbf{b} = \mu_0(\mathbf{h} + \mathbf{m} + \mathbf{m}^e), \tag{1.1b}$$

$$\mathbf{F} = \operatorname{Grad} \chi, \quad \operatorname{div} \boldsymbol{\sigma}_{\text{tot}} = \mathbf{f}^e, \quad \boldsymbol{\sigma}_{\text{tot}} = \boldsymbol{\sigma}_{\text{tot}}^T. \tag{1.1c}$$

Here, what the symbols stand for is as follows: $\mathbf{e}$ – electric field, $\mathbf{d}$ – electric displacement, $\mathbf{p}$ (resp. $\mathbf{p}^e$) – intrinsic (resp. external) polarization, ρ^e – external charge density; $\mathbf{h}$ – magnetic field, $\mathbf{b}$ – magnetic flux, $\mathbf{m}$ (resp. $\mathbf{m}^e$) – intrinsic (resp. external) magnetization; χ – deformation, $\mathbf{F}$ – deformation gradient, $\boldsymbol{\sigma}_{\text{tot}}$ – total stress, $\mathbf{f}^e$ – external body force; ϵ_0 (resp. μ_0) – electric permittivity (resp. magnetic permeability) of vacuum. The classic uncoupled theories of electrostatics, magnetostatics and elasticity for continuum media are completed by providing constitutive laws such as

$$\mathbf{d} = \mathbf{d}(\mathbf{p}), \quad \mathbf{b} = \mathbf{b}(\mathbf{m}), \quad \boldsymbol{\sigma}_{\text{tot}} = \boldsymbol{\sigma}_{\text{tot}}(\mathbf{F}).$$

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In the era of nanotechnology and bioengineering, much of attention has been focused on multifunctional or multiferroic materials where strain, polarization and magnetization are simultaneously coupled. These materials have broad applications ranging from the technologies of actuators, sensors, imaging devices to smart self-adaptive structures, artificial muscles, etc. (Fiebig, 2005; Nan et al., 2008). Further, to address novel phenomena observed in complex and heterogeneous systems such as granular media and composites, it is often necessary to consider nonlocal effects that motivate gradient theories. These issues have been addressed in thematic topics of strain gradient theory (Fleck et al., 1994), polarization gradient theory (Mindlin, 1968; Buchanan et al., 1989), flexoelectricity among others (Tagantsev, 1986; Majdoub et al., 2008). The literature, however, lacks of a unified treatment that considers general magneto-electro-elastic materials and their gradient effects.

Since the seminal definitive work of Toupin (1956, 1960) that will be briefly reviewed in Section 2, the key ingredients of a theory for electro-elastic materials have been well understood and are worthwhile to mention. First, a somewhat peculiar stress term, namely the Maxwell stress, emerges from electrostatic field such that the elastic state and electric state of the body are intrinsically coupled. As a nonlinear function of electric field, the Maxwell stress gives rise to substantial difficulty in solving a generic boundary value problem concerning an electro-elastic body. A second important issue, as always, lies in the formulating generic nonlinear constitutive laws that relate elastic, electric and magnetic quantities and guarantees reasonable physical behaviors. Toupin (1956) and many subsequent authors (Eringen, 1963; Mindlin, 1968; McMeeking and Landis, 2005; Suo et al., 2008; Tian, 2007; Tian et al., 2012) propose such constitutive relations by postulating a special form of the stored (or internal) energy function, and then systematically restrict the form of the internal energy function, expand and truncate the internal energy function that will eventually yield simple, possibly linear, constitutive relations. Toupin (1956) in fact began his theory by deriving the basic field equations from the Maxwell equations and mechanical balance laws, and then formulated the constitutive relations by postulating a stored energy function and showed the equivalence between the field equations and the *principle of virtual work*.

Nowadays, as we understand that the *principle of virtual work* (*or power*) can in general be regarded as the weak form of a variational principle, it is of significant interest to have a variational formulation for magneto-electro-elastic bodies based on the physical free energy. Since our main goal is to identify the static equilibrium of the body at a constant temperature, for clarity we will assume that the body is at a constant temperature, has constant entropy, and hence the equilibrium state of the body is dictated by the *principle of minimum free energy* (Gibbs, 1878, p. 109; Ericksen, 1991, Chapter 1). For a continuum body, the employment of the *principle of minimum free energy* requires two critical hypotheses: (i) the set of thermodynamic variables that completely describe the state of the continuum body and their possible variations, and (ii) the total free energy of the body when the body is interacting with well-defined boundary devices. It is fruitless to justify these hypotheses in the framework of continuum theory, though atomistic models may shed light on the foundation of these hypotheses.

We therefore begin our energy formulation for a magneto-electro-elastic body with these two hypotheses whereas the rest of the theory is mathematically deducted. Also, the Maxwell equations (1.1a) and (1.1b) are taken as premises but the concept of stresses including the Maxwell stress are regarded as derived notions. The advantages of such a variational formulation based on the principle of minimum free energy include: (i) embracing the framework of Gibbs, the proposed energy formulation admits clear thermodynamic interpretation and may even be proved by the Second Law in a proper setup (Fosdick and Tang, 2007); (ii) we no longer need to forcefully separate the total stress into the *local* mechanical stress and the *nonlocal* Maxwell stress. The Maxwell stress will emerge naturally from the first variation of the total free energy. This calculation also explains the relation between different forms of Maxwell stress in the literature; (iii) novel physical phenomena such as the gradient effects of strain, polarization and magnetization can be treated uniformly without introducing additional constitutive relations. All constitutive assumptions will be lumped into the form of stored/internal energy function of the material; and finally but not the least important, (iv) in regard of recent development of $\Gamma-$convergence and homogenization, the proposed energy formulation can be directly applied to rigorously *derive* a hierarchy of theories at different scaling limits (Tian, 2007) and for lower dimensional bodies (Section 6.3), and to obtain bounds on the effective properties of composites (Milton, 2002; Liu, 2013).

In addition, we notice that the proposed energy formulation is closely related with the phenomenological theories of Landau (Landau and Lifshitz, 1935; Landau, 1937; Ginzburg and Landau, 1950) concerning phase transitions of ferromagnetic, ferroelectric and superconducting materials, and the theory of austenite–martensite phase transition (James and Wuttig, 1998). The formulation is also closely related with the Hashin–Shtrikman's variational principle that is important for finding bounds on the effective properties of composites (Hashin and Shtrikman, 1962). As discussed in Liu (2013), the field equations (1.1) in general may enjoy many different variational formulations whose Euler–Lagrange equations are consistent with the field equations (1.1); neither the energy functional nor the independent state variables have to be the same. However, for evolution problems[1] and stability analysis (Chen, 2009; Xu et al., 2010; Dorfmann and Ogden, 2010), we have to identify the physical free energy and possible variations of state variables based on the physical ground. A wrong choice may yield erroneous or even opposite results (Liu, 2013, Section 6; Bustamante and Merodio, 2012).

Three solutions of the proposed formulation are presented in Section 6. We highlight here a few interesting results and potential applications. In Section 6.1 we address the equilibrium shape of a soft nonlinear elastic ellipsoid in an applied

[1] For such evolution problems, a phenomenological kinetic law, e.g., velocity $\propto$ driving force, has to be postulated to close the system, see e.g., Abeyaratne and Knowles (2006, p. 50).