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# Generation of an arbitrary nondiffractive optical field with upper bound diffraction efficiency: Theory and experimental generation of Parabolic fields

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ABSTRACT

This paper theoretically demonstrates that the Fourier spectra of an arbitrary nondiffractive optical field and the corresponding coding noise do not overlap when the kinoform is used to encode the field. Consequently, to generate the desired nondiffractive optical field with upper bound diffraction efficiency, we remove the coding noise at the Fourier plane with a binary filter and apply inverse Fourier transform. To illustrate our theoretical approach, we generate, for the first time, both numerically and experimentally Parabolic optical fields with upper bound diffraction efficiency.





## 1. Introduction

Nondiffractive optical fields have attracted considerable attention because of their unique property, i.e., propagation invariance over a finite distance [1–3]. They are used for example in the generation of photonic nonlinear structures, periodic and quasi-periodic intensity distributions, and vector optical fields [4–6].

Usually Phase holograms (PH) are used to generate complex optical fields, because they possess high diffraction efficiency and are accurately implemented using liquid–crystal (LC) spatial light modulator (SLM) [7,8]. The existence of the upper bound diffraction efficiency was first recognized by Wyrowski [9]. In this way, the generation of nondiffractive optical fields with upper bound diffraction efficiency is shown in [10–14]. In those cases, the phase of the transmittance of the PH equals the phase of the desired nondiffractive optical field, i.e., the nondiffractive optical fields are encoded using their corresponding *kinoforms*. Additionally, in the transmittance Fourier domain, the spectrum of the encoded field is noise free. The generation of an arbitrary nondiffractive Bessel beam using its phase modulation was analytically and experimentally verified in [10]. Using numerical results, in [11,12], the authors show that the class the periodic and quasiperiodic nondiffractive fields composed by $Q$ planes waves of constant amplitudes and uniform phases can be generated employing only the phase information [11,12]. Recently, the results [10–12] were theoretically unified in [14]. On the other hand, experimental generation of Mathieu beams with upper bound diffraction efficiency is addressed in [13]. Unfortunately, the results [13] were not analytically justified. Using blazed phase computer-generated holograms (CGH), the generation of nondiffractive Parabolic beams was addressed in [15]. However, employing those phase holograms, the maximum diffraction efficiency obtained into the $\pm 1$ orders is 33.9% and they do not satisfy the upper bound diffraction efficiency.

The main goal of this paper is to extend the previous results [10–12,14] and consider the generation of an arbitrary nondiffractive optical field with upper bound diffraction efficiency. To do that, we theoretically prove that, in the kinoform Fourier spectrum, the spectra of the nondiffractive field and coding noise do not overlap. This goal is also motivated by the fact that nondiffractive fields also find applications in optical tweezers [16], where the diffraction efficiency plays a key role [17]. As an experimental application of the proposed approach, we demonstrate for the first time the generation of nondiffractive Parabolic fields with upper bound diffraction efficiency.

## 2. Kinform Fourier spectrum of an arbitrary nondiffractive field

This section focusses on the computation of the kinoform Fourier spectrum of an arbitrary nondiffractive field. Our analyzes

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use cylindrical coordinates. We are interested in nondiffractive optical fields $s(r,\theta)$ expressed by means of Whittaker integral [18],

$$s(r,\theta)=\int_{-\pi}^{\pi} A(\varphi)\exp\left[-ikr\cos(\theta-\varphi)\right]d\varphi, \tag{1}$$

which can be viewed as the inverse Fourier transform in cylindrical coordinates of $S(\rho,\varphi)=A(\varphi)\delta(2\pi\rho-k)$, where $k$ is the wavenumber, $\delta(\cdot)$ is the Dirac delta, and $A(\varphi)$ is an arbitrary complex function. This means that the Fourier spectrum of the complex field is an annular delta of radius $k/2\pi$, which is modulated by the angular spectrum $A(\varphi)$.

In addition, we assume that there exists a complete orthogonal basis $\psi_n(r,\theta)$, for all $n$, such that any field defined in (1) is expressed as a linear combination of $\psi_n(r,\theta)$,

$$s(r,\theta)=\sum_{n=-\infty}^{\infty} a_n\psi_n(r,\theta), \tag{2}$$

where $a_n$, for all $n$, are complex coefficients. It is worth to mention that the complete basis is not unique. Some examples of orthogonal bases are Bessel fields [1], Elliptic fields [2], and Parabolic fields [3].[1] For a fixed field $s(r,\theta)$, we select $\psi_n(r,\theta)$, for all $n$, as Bessel fields, i.e., $\psi_n(r,\theta)=J_n(kr)\exp(in\theta)$, where $J_n(\cdot)$ is the Bessel function of order $n$. For instance, the Jacobi–Anger identity establishes that a plane wave is expressed by superposition of Bessel fields. Specifically, we have that $\exp(iky)=\sum_{m=-\infty}^{\infty}J_n(kr)\exp(in\theta)$, where $y$ is the cartesian coordinate.

Now we define the kinoform. Considering $s(r,\theta)=|s(r,\theta)|\exp[i\phi(r,\theta)]$, the *kinoform* of the complex field is defined as $h(r,\theta)=\exp[i\phi(r,\theta)]$, i.e., it is obtained by removing the magnitude $|s(r,\theta)|$ from $s(r,\theta)$. In order to assist the presentation of the contribution, we recall the following property that provides the necessary and sufficient condition for the optical fields to be encoded with upper bound diffraction efficiency. The proof is found in [14].

**Property 1.** The complex optical field $s(r,\theta)$ satisfies

$$h(r,\theta)=\beta_{ub}s(r,\theta)+\varepsilon(r,\theta), \tag{3}$$

$$\int_{-\pi}^{\pi}\int_{0}^{R} s^*(r,\theta)\varepsilon(r,\theta)r\,dr\,d\theta=0, \tag{4}$$

where $\varepsilon(r,\theta)$ is the coding noise and R is the radius of the circular pupil, which limits the optical field, if and only if the upper bound amplitude gain $\beta_{ub}$ is expressed as

$$\beta_{ub}=\frac{\int_{-\pi}^{\pi}\int_0^R |s(r,\theta)|r\,dr\,d\theta}{\int_{-\pi}^{\pi}\int_0^R |s(r,\theta)|^2\,r\,dr\,d\theta}. \tag{5}$$

Using (3) and (4), the total light intensity of the phase hologram $I_h$ is given by $I_h=\beta_{ub}^2 I_s+I_\varepsilon$, where $\beta_{ub}^2 I_s$ and $I_\varepsilon$ are, respectively, the total light intensities of the encoded field and coding noise. The light intensities $I_s$ and $I_\varepsilon$ can be computed as $I_s=\int_{-\pi}^{\pi}\int_0^R|s(r,\theta)|^2\,r\,dr\,d\theta$ and $I_\varepsilon=\int_{-\pi}^{\pi}\int_0^R|\varepsilon(r,\theta)|^2\,r\,dr\,d\theta$. By removing the coding noise term from the kinoform, the optical field $s(r,\theta)$ is generated with the upper bound diffraction efficiency given by $\eta_{ub}=\beta_{ub}^2 I_s/I_h$. This says that the generation of the optical field uses the upper limit percentage of light.

In the context of digital holography, the coding noise is removed at the Fourier plane if the spectrum of the encoded nondiffractive field is noise free [10–12,14]. However, observe that Property 1 does not involves any Fourier information. Next property introduces the main contribution of this paper, i.e., we demonstrate that the Fourier spectra of an arbitrary nondiffractive field and coding noise do not overlap.

**Property 2.** Consider Property 1 with R being infinity and let $s(r,\theta)$ be an arbitrary nondiffractive complex optical field defined in (2), then the domains of $S(\rho,\varphi)$ and $E(\rho,\varphi)$ are disjoints, where $S(\rho,\varphi)$ and $E(\rho,\varphi)$ are, respectively, the Fourier transforms of the field $s(r,\theta)$ and coding noise $\varepsilon(r,\theta)$.

**Proof.** First, using (2) and (5), the upper bound amplitude gain for an arbitrary nondiffractive field $s(r,\theta)$ is expressed as

$$\beta_{ub}=\frac{1}{2\pi}\frac{\int_{-\pi}^{\pi}\int_0^R|s(r,\theta)|r\,dr\,d\theta}{\sum_{n=-\infty}^{\infty}|a_n|^2\int_0^R J_n^2(kr)r\,dr}. \tag{6}$$

Now, substituting (2) into $|s(r,\theta)|=s^*(r,\theta)h(r,\theta)$ and integrating, the following relation holds

$$\int_{-\pi}^{\pi}\int_0^R|s(r,\theta)|r\,dr\,d\theta=\sum_{n=-\infty}^{\infty}a_n^*\int_{-\pi}^{\pi}\int_0^R h(r,\theta)J_n(kr)\exp(-in\theta)r\,dr\,d\theta. \tag{7}$$

Second, the Fourier expansion of the kinoform $h(r,\theta)$ is given by

$$h(r,\theta)=\sum_{n=-\infty}^{\infty}C_n(r)\exp(in\theta), \tag{8}$$

where $C_n(r)$, for all $n$, are the Fourier coefficients. Additionally, we expand the coefficient $C_n(r)$ using Fourier–Bessel series, i.e.,

$$C_n(r)=\sum_{q=1}^{\infty}c_{q,n}J_n(\lambda_{q,n}r), \tag{9}$$

where $\lambda_{q,n}$, for $q=1,\ldots,\infty$, are the roots of $J_n(r)+U_n r J_n'(r)=0$, where $J_n'(r)$ is the first derivative of $J_n(r)$, $U_n$ is a real-valued constant, and the coefficient $c_{q,n}$ is given by [19]

$$c_{q,n}=\frac{1}{\int_0^R J_n^2(\lambda_{q,n}r)r\,dr}\int_0^R C_n(r)J_n(\lambda_{q,n}r)r\,dr. \tag{10}$$

We select the constant $U_n$ such that there exist roots $\lambda_{q_0^n,n}=k$, for some integer $q_0^n$ and all $n$, which satisfy $J_n(k)+U_n kJ_n'(k)=0$. All roots $\lambda_{q,n}$, for $q\neq q_0^n$ and for all $n$, are not equal to $k$ [19]. Thus, Eq. (8) becomes

$$h(r,\theta)=\sum_{n=-\infty}^{\infty}c_nJ_n(kr)\exp(in\theta)+\varepsilon(r,\theta), \tag{11}$$

where

$$\varepsilon(r,\theta)=\sum_{n=-\infty}^{\infty}\sum_{\substack{q=1\\ q\neq q_0^n}}^{\infty}c_{q,n}J_n(\lambda_{q,n}r)\exp(in\theta), \tag{12}$$

and the coefficient $c_n$, according to the expansions (8) and (10), is expressed as

$$c_n=c_{q_0^n,n}=\frac{1}{2\pi\int_0^R J_n^2(kr)r\,dr}\int_{-\pi}^{\pi}\int_0^R h(r,\theta)J_n(kr)\exp(-in\theta)r\,dr\,d\theta. \tag{13}$$

Combining (6), (7), and (13), the coefficient $c_n$ can be rewritten as $c_n=\beta_{ub}a_n$. Furthermore, from Property 1 and (11), the coding noise is defined by (12). Finally, considering $R$ infinity, we show that the Fourier transforms of $s(r,\theta)$ and $\varepsilon(r,\theta)$ do not overlap, i.e., $S^*(\rho,\varphi)E(\rho,\varphi)=0$. The Fourier transform of the nondiffractive field $s(r,\theta)$ is an annular delta, which is modulated by the complex function $A(\varphi)$ (see Whittaker integral in (1)). On the other hand, the Fourier transform of the coding noise consists of the superposition of infinite number of annular deltas. The radius of each annular delta is $\lambda_{q,n}/2\pi$ and is not equal to $k/2\pi$. Therefore, the domains of $S(\rho,\varphi)$ and $E(\rho,\varphi)$ are disjoints. □

Property 2 says that for an arbitrary nondiffractive optical field the spectra of the field and coding noise do not overlap. In other words, the Fourier transform of the coding noise evaluated at

[1] The authors in [3] pointed out that the corresponding separation constant is continuous for Parabolic fields. However, to obtain orthogonal fields, the separation constant must be discrete.

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