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Approximate solutions to Mathieu's equation

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A B S T R A C T

Mathieu's equation has many applications throughout theoretical physics. It is especially important to the theory of Josephson junctions, where it is equivalent to Schrödinger's equation. Mathieu's equation can be easily solved numerically, however there exists no closed-form analytic solution. Here we collect various approximations which appear throughout the physics and mathematics literature and examine their accuracy and regimes of applicability. Particular attention is paid to quantities relevant to the physics of Josephson junctions, but the arguments and notation are kept general so as to be of use to the broader physics community.

1. Introduction

Mathieu's equation,

$$\frac{d^2\psi}{dz^2} + (a - 2\eta\cos(2z))\psi = 0. \tag{1}$$

has appeared in theoretical physics in many different contexts. Mathieu originally formulated the equation to describe the vibration modes of an elliptical membrane [1], but the equation has since been applied to the theory of quadrupole ion traps [2–4], ultracold atoms [5] and quantum rotor models [6,7]. This equation has also found attention as a simplified model of a particle moving in a periodic potential [8].

Although Mathieu's equation is easy to solve numerically, and although exact results are achievable in certain limits, a general analytic solution of Mathieu's equation has not yet been achieved. Instead, there exists throughout the literature, both on physics and mathematics, a myriad of approximations and numerical methods which may be used to extract quantities of interest. It is the goal of this paper to collect these approximations together in one place for easy reference, to review them explicitly and explore their regimes of validity. The focus is to illustrate and compare the results found in the vast body of literature on this topic.

This manuscript will focus primarily on applications of Mathieu's equation to the physics of Josephson junctions [9–12], however we will keep the notation general as the results presented herein may be of use across diverse fields. Josephson junctions are elements in superconducting circuits, which are of great interest due to potential applications in quantum technology [9,13,14].

A single Josephson junction is governed by the Hamiltonian

$$H = -4E_C\frac{\partial^2}{\partial\phi^2} - E_J\cos(\phi) \tag{2}$$

where $E_C = e^2/2C$ is the charging energy, C is the junction capacitance, E_J the Josephson energy and ϕ is the phase difference of the superconducting condensate across the junction. With this Hamiltonian, the time-independent Schödinger equation becomes

$$\left[-4E_C\frac{\partial^2}{\partial\phi^2} - E_J\cos(\phi)\right]\psi = E\psi. \tag{3}$$

This reduces to Mathieu's equation upon making the substitutions $\phi/2 \to z$, $E/E_C \to a$, $E_J/2E_C \to \eta$. To maintain generality, we will retain the notation of Mathieu's equations, but we will bear these substitutions in mind and make frequent reference to results obtained in the theory of Josephson junctions.

The focus will be on quantities corresponding to physical observables in Josephson junctions. We will therefore not be concerned with the details of the Mathieu functions themselves (physically, the wavefunctions of the Josephson junction array), but primarily on the characteristic value a, the floquet exponent ν, and related quantities depicted in Fig. 1.

Each of these quantities will be discussed in detail below, but each can be understood loosely as follows: $t = b_1 - a_0$ is the difference between the lowest characteristic value of an odd-parity Mathieu function and the lowest characteristic value of an even-parity Mathieu func-

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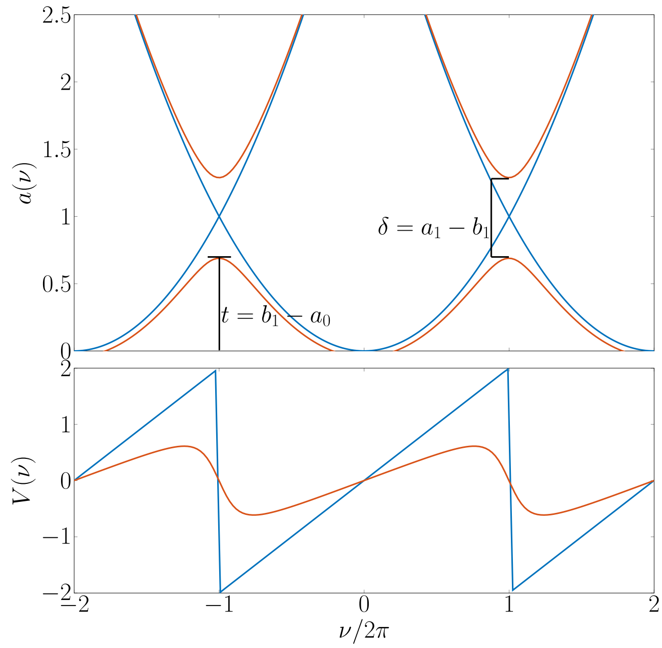

Fig. 1. The characteristic value $a(\nu)$ and its derivative with respect to the Floquet exponent ν for Mathieu's equation with $\eta = 0$ (blue) and $\eta = 0.3$ (red). (For interpretation of the references to colour in this figure legend, the reader is referred to the Web version of this article.)

tion. Physically it corresponds to the bandwidth of the lowest energy band of a Josephson junction.

For characteristic values between a_1 and b_1, stable Mathieu functions do not exist. $\delta = a_1 - b_1$ represents a gap in characteristic values of stable Mathieu functions. Physically, δ corresponds to the band gap in the energy spectrum of the Josephson junction.

$V(\nu) = \mathrm{d}a/\mathrm{d}\nu$ is a quantity little discussed in the mathematics literature, but in the physics of Josephson junctions it is known as the effective voltage [9].

In experiments on Josephson junctions the quantity η is often a controlled parameter. In fact, if one adopts a SQUID geometry, E_J, and by extension η, can be tuned in real time by adjusting the applied magnetic flux [15]. We are therefore primarily interested with how these various parameters vary with η. In Fig. 1 we have ploted $a(\nu)$ and $V(\nu)$ for $\eta = 0$ and $\eta = 0.2$.

The limits of both strong coupling ($\eta \gg 1$) and weak coupling ($\eta \ll 1$) are relatively straightforward. In both cases the characteristic values can be expressed as asymptotic expansions in powers of η or $1/\eta$ respectively. Below we will explore both of these extreme limits of the model, and investigate the region $\eta \sim 1$ where the approximations are expected to break down. We will also examine properties of Mathieu's equation which may be deduced from periodicity arguments, as these are expected to be valid for any value of η.

2. Small η

In the limit that $\eta \to 0$, Mathieu's equation becomes

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\phi^2} + a\psi = 0. \tag{4}$$

This differs from Schrödinger's equation for a particle moving in free space only in that the co-ordinate ϕ has the topology of a circle. In this limit, the eigenvalues are continuous and do not form separate energy bands or levels. The Mathieu functions themselves are simply $\pm\cos(\sqrt{a_n}z)$, $\pm\sin(\sqrt{b_{n+1}}z)$ (as can be trivially verified). By convention we take the sign to be positive. The characteristic value of the sine solution is denoted b_{n+1} rather than a_n by convention and for later convenience, but it should be interpreted the same way (physically, as an energy eigenvalue).

For finite η corrections must be added to the simple cos and sin solutions, however the solutions retain their periodicity and parity. The finite η generalisations are referred to as cosine-elliptic or sine-elliptic functions respectively, and are denoted $\mathrm{ce}_n(z,\eta)$ and $\mathrm{se}_{n+1}(z,\eta)$. These can generally not be expressed in closed form. However, we can obtain many physically relevant quantities without direct reference to these functions.

At $\eta = 0$, stable solutions exist for any value of a_n (or b_m). However, at finite η band gaps appear, and solutions are only stable when the characteristic value a is $a_n \le a \le b_{n+1}$, where n is an integer and where we have used a without a subscript to denote an arbitrary characteristic number which will generally be of fractional order.

Physically, this stability/instability of solutions manifests itself in the form of energy bands, so that the stability diagram of Mathieu's equation gives us the band structure of a Josephson junction. At a given value of η, the characteristic energy is a periodic function of the characteristic exponent ν (to be introduced below). Many quantities of physical interest can be expressed in terms of the lowest and highest energies in a band, a_n and b_{n+1} respectively. For example, the ground state bandwidth is just $b_1 - a_0$, and the gap between the ground and first excited state is $a_1 - b_1$.

At small η, the characteristic values can be expanded in powers of η [16], giving

$$\begin{aligned} a_0 &= -\frac{1}{2}\eta^2 + \frac{7}{128}\eta^4 - \frac{29}{2304}\eta^6 + \frac{68687}{18874368}\eta^8 + \mathcal{O}(\eta^{10}) \\ b_1 &= 1 - \eta - \frac{1}{8}\eta^2 + \frac{1}{64}\eta^3 - \frac{1}{1536}\eta^4 - \frac{11}{36864}\eta^5 + \frac{49}{589824}\eta^6 \\ &\quad - \frac{55}{9437184}\eta^7 - \frac{265}{113246208}\eta^8 + \mathcal{O}(\eta^9). \end{aligned} \tag{5}$$

The expression for a_1 is identical to b_1, but with $\eta \to -\eta$. Similar expansions for higher order characteristic values can be found in section 2.151 of ref. [16].

3. Large η

When $\eta \gg 1$, z remains close to the minima of $\cos 2z$, so that when expanded as a Taylor series only the second order term is relevant. This reduces the Mathieu equation to the form of Schrödinger's equation for a harmonic oscillator, so that the Mathieu functions may be approximated by the wavefunctions of a harmonic oscillator

$$\psi_n^{\mathrm{HO}}(z) = c_n H_n\left((2\eta)^{1/4}z\right) e^{-\frac{1}{2}\sqrt{2\eta}z^2} \tag{6}$$

with energy levels

$$a_n = 4\sqrt{\eta}(n + \frac{1}{2}) - 2\eta \tag{7}$$

where $H_n(x)$ are Hermite polynomials familiar from the theory of the quantum harmonic oscillator, c_n is a normalization constant and the constant shift 2η comes from the expansion of the cosine. Introducing $x = (2\eta)^{1/4}z$ this simplifies to

$$\psi_n^{\mathrm{HO}}(x) = c_n H_n(x) e^{-\frac{1}{2}x^2}. \tag{8}$$

In this limit, the Mathieu equation can be interpreted as the Hamiltonian for a tight-binding model [17]. Following the standard textbook analysis of the tight-binding model, we can calculate the bandwidth of the characteristic values of the Mathieu equation via

$$b_1 - a_0 = -\int \mathrm{d}z\psi(z)\psi(z-\pi)V(z) \tag{9}$$

where $V(z) = -2\eta(1+\cos(z)) \approx \eta z^2 + \text{const.}$ and we have shifted our integration variable $2z \to z$. A detailed calculation of this integral, along