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Connection between wave transport through disordered 1D waveguides and energy density inside the sample: A maximum-entropy approach

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ABSTRACT

We study the average energy – or particle – density of waves inside disordered 1D multiply-scattering media. We extend the transfer-matrix technique that was used in the past for the calculation of the intensity beyond the sample to study the intensity in the interior of the sample by considering the transfer matrices of the two segments that form the entire waveguide. The statistical properties of the two disordered segments are found using a maximum-entropy ansatz subject to appropriate constraints. The theoretical expressions are shown to be in excellent agreement with 1D transfer-matrix simulations.

1. Introduction

Studies of the scaling of the average transmission and the full probability distribution of transmission and conductance through disordered media at various degrees of spatial averaging over the output and input surface have played a central role in localization and mesoscopic physics [1–6]. Interest in the profiles of energy or particle density inside random open samples, which has been of great current interest [7–12], goes back even further. In the work of Ref. [13] a detailed calculation is reported in which successive averaging over impurities is performed to yield the result of Eq. (26) of that reference for the average intensity inside the sample. A further generalization of this result (allowing for impedance mismatch at the generator and the load, in the electric-circuit terminology used by the authors), which makes use of the notion of transfer matrices, is given in Ref. [14].

It is interesting to recall that for a classical system, the intensity $I(z)$ in the interior of a diffusing sample falls linearly within the sample, as required by Fick's law of particle diffusion; this yields a constant diffusion coefficient D_0. A first-order correction to the diffusion equation due to localization effects was obtained recently by introducing a one-loop weak localization correction together with the assumption of the self-consistency of this approximation [15,16]. This has yielded a generalized diffusion equation with a position-dependent diffusion coefficient, $D(x)$. This diffusion coefficient is increasingly renormalized with increasing depth into the sample by the destructive interference of waves returning to points inside the medium. Good agreement with this self-consistent theory [17] is obtained in simulations and in optical measurements in a multichannel slot in a periodic 2D structure. When extended to the time domain [10], the self-consistent theory gives good agreement [18] with the transmitted pulse profile for ultrasound in the localization transition. For more deeply localized samples, however, microwave transmission at long times was dramatically slowed down because of the increasing contribution of long-lived quasi-normal modes to transmission at late times [19]. An exact theory beyond the single-loop approximation has been developed based on the supersymmetry approach [20]. The problem of an inhomogeneous diffusion coefficient dependent on system size has been discussed recently in Ref. [21], based on the results of Ref. [13].

An alternative approach to the statistical properties of disordered conductors based on a *maximum-entropy ansatz* was presented in detail in Ref. [22], where an equation governing these properties, the Dorokhov–Mello–Pereyra–Kumar (DMPK) equation, is derived using this method. In this approach, the disordered system is assumed to contain a large number of very weak scatterers: this was called the *dense-weak-scattering limit*. It is expected that in this limit the maximum-entropy ansatz will give results largely independent of the microscopic details. The *transfer matrix* is particularly useful in this context due to its multiplicative property. The result is thus a random-matrix theory (RMT) of disordered systems, in which one studies an ensemble of transfer matrices.

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To the best of our knowledge, RMT has not been widely used to explore the statistics of propagation inside random media. One exception is Ref. [14] which, in the limit of uncorrelated disorder, finds an exact solution to the problem. Ref. [13] uses, instead, recursive relations and an induction method. Ref. [21] makes use of the results of Ref. [13] to compute the average intensity inside the sample.

In the present contribution, we develop an approach based on the DMPK equation mentioned above to find the average intensity profile inside random 1D samples. As we indicated, the method based on a maximum-entropy ansatz, although not exact, is expected to give results largely independent of the microscopic details. These calculations are in excellent agreement with computer simulations. In our opinion, the interest of the present point of view lies in the conceptual simplicity of the maximum-entropy approach, and the possibility of extending the analysis to other quantities of physical interest; it may provide an opening to calculate the average profile of energy density inside 1D or quasi-1D samples for transmission eigenchannels with specified values of transmission. These profiles were recently found in computer simulations [12]. The profiles were found to have a form related to the auxiliary localization lengths proposed by Dorokhov [5]. The structure of these profiles is consistent with the generalized diffusion equation with a position dependent diffusion coefficient and appropriately chosen source term and boundary conditions [12]. Since it is possible to manipulate the incident profiles of classical waves, the prospect exists of controlling the energy density inside random systems.

The paper is organized as follows. In Section 2 we construct an expression for the intensity inside a 1D sample, when incidence is from the left of the sample. Using the maximum-entropy ansatz outlined above, we average this result over an ensemble of disordered configurations to obtain our central result, Eq. (11a) below. Excellent agreement is found in a comparison of these results with computer simulations. Discussion of these results and perspectives for future research are presented in Section 3. In order not to interrupt the flow of the presentation in the main text, two appendices containing mathematical details have been added.

2. The intensity inside a 1D waveguide

Consider the problem of scattering by a one-dimensional (1D) random distribution of scatterers, as illustrated in Fig. 1. This situation may arise in a Quantum Mechanical (QM) problem describing electronic scattering in a disordered conductor, or, more generally, in a wave-scattering problem in a disordered waveguide supporting a single transverse mode, or for a plane wave impinging upon a random layered medium. In what follows, we shall refer specifically to the first type of problem and use the QM nomenclature, although the notions from both fields can be used interchangeably.

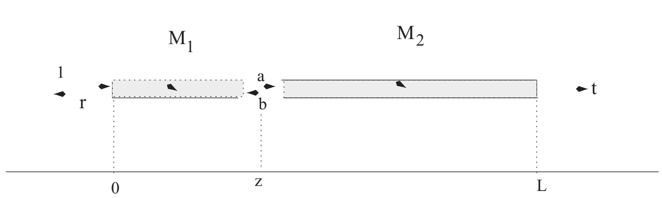

Fig. 1. The scattering problem associated with the 1D disordered waveguide described in the text. The waveguide has length L and can support one propagating mode, or channel. Indicated are the amplitudes of the incident, transmitted and reflected waves at either end of the waveguide. A small gap is opened at the point z inside the sample, where the amplitudes of the waves travelling to the right and left are also shown. The transfer matrices M_1, M_2 of the two parts of the sample are also indicated.

The amplitude of the plane wave incident from the left is taken to be 1; the effect of the scattering process is to produce a reflected wave with amplitude r on the left of the whole system, and a transmitted wave on the right, with amplitude t.

Inside the conductor, a distance z from the left side of the sample, the wave function consists of a wave travelling to the right, with amplitude a, and a wave travelling to the left, with amplitude b, as also shown in Fig. 1. As described in the Introduction, the goal is to find the average intensity of the wave at the point z inside the conductor.

We express the transfer matrices of the two portions of the waveguide as

$$M_i = \begin{bmatrix} \alpha_i & \beta_i \\ \beta_i^* & \alpha_i^* \end{bmatrix}, \quad i = 1, 2, \tag{1}$$

with the condition $|\alpha_i|^2 - |\beta_i|^2 = 1$, thus satisfying the requirements of time-reversal invariance and flux conservation. When no index i is employed, we shall understand the various quantities to refer to the wire as a whole.

From the definition of the transfer matrix, we have

$$M_2 \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}. \tag{2}$$

We invert this equation to find a and b, making use of the relation

$$M_2^{-1} = \begin{bmatrix} \alpha_2^* & -\beta_2 \\ -\beta_2^* & \alpha_2 \end{bmatrix}, \tag{3}$$

to find

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} t\alpha_2^* \\ -t\beta_2^* \end{bmatrix}, \tag{4}$$

where the transmission amplitude t can be expressed as

$$t = \frac{1}{\alpha^*} = \frac{1}{\alpha_2^*\alpha_1^* + \beta_2^*\beta_1}. \tag{5}$$

The intensity inside the gap is then given by the equivalent expressions

$$I_z(M_1, M_2) = |ae^{ikz} + be^{-ikz}|^2 \tag{6a}$$

$$= T|\alpha_2^* e^{ikz} - \beta_2^* e^{-ikz}|^2 \equiv TF_z(M_2) \tag{6b}$$

$$= \frac{|\alpha_2^* e^{ikz} - \beta_2^* e^{-ikz}|^2}{|\alpha_2\alpha_1 + \beta_2\beta_1^*|^2}. \tag{6c}$$

Using the polar representation defined in Appendix A.1, we can write the functions appearing in Eqs. (6) as

$$F_z(M_2) = 1 + 2\lambda_2 - 2\sqrt{\lambda_2(1+\lambda_2)}\cos(2(\mu_2 - \theta_2 + kz)), \tag{7a}$$

$$1/T = 1 + \lambda_1 + \lambda_2 + 2\lambda_1\lambda_2 + 2\sqrt{\lambda_1\lambda_2(1+\lambda_1)(1+\lambda_2)}\cos(2(\theta_2 + \mu_1 - \mu_2)). \tag{7b}$$

The above expressions refer to one sample, i.e., to one configuration of disorder. Making the assumption of uncorrelated disorder, the various quantities referring to the two sections that form the full sample are statistically independent of one another.

The average over an ensemble of configurations of the intensity $I_z(M_1, M_2)$ may be computed using the probability distribution of the transfer matrices for the two sections of the waveguide, i.e.,

$$\langle I_z(M_1, M_2)\rangle = \iint I_z(M_1, M_2) p_z(M_1) p_{L-z}(M_2)\, d\mu(M_1)\, d\mu(M_2). \tag{8}$$