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Regularized variational theories of fracture: A unified approach

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A B S T R A C T

The fracture pattern in stressed bodies is defined through the minimization of a two-field pseudo-spatial-dependent functional, with a structure similar to that proposed by Bourdin–Francfort–Marigo (2000) as a regularized approximation of a parent free-discontinuity problem, but now considered as an autonomous model *per se*. Here, this formulation is altered by combining it with structured deformation theory, to model that when the material microstructure is loosened and damaged, peculiar inelastic (structured) deformations may occur in the representative volume element at the price of surface energy consumption. This approach unifies various theories of failure because, by simply varying the form of the class for admissible structured deformations, different-in-type responses can be captured, incorporating the idea of cleavage, deviatoric, combined cleavage-deviatoric and masonry-like fractures. Remarkably, this latter formulation rigorously avoid material overlapping in the cracked zones. The model is numerically implemented using a standard finite-element discretization and adopts an alternate minimization algorithm, adding an inequality constraint to impose crack irreversibility (*fixed* crack model). Numerical experiments for some paradigmatic examples are presented and compared for various possible versions of the model.

1. Introduction

In a fundamental paper, Francfort and Marigo (1998) first introduced a variational approach to fracture through the minimization of an energy functional composed of a bulk term, i.e., the strain of the sound material, and a surface energy term *à la* Griffith, and by adding proper irreversibility conditions for crack opening. Later on the same authors together with Bourdin et al. (2000) proposed, essentially for numerical purposes, a variational approximation of this difficult *free-discontinuity problem* with a regularized elliptic two-field functional, where one field is representative of the macroscopic displacement in the body, while the other one, here denoted by s, is similar to an isotropic-damage parameter because it varies between 0 and 1 and takes the null value in a cracked (damaged) zone and 1 away from it.[1] This latter formulation leads to a pseudo spatial-dependent theory since it allows for spatial gradients of s to affect the value of the stored energy functional. The regularized functional is characterized by a parameter here indicated by l: by extending the Ambrosio and Tortorelli (1990) weak formulation of the Mumford–Shah functional in problems of image-segmentation, it is proved in Bourdin et al. (2000) for the case of antiplane shear that as $l \to 0$, the elliptic functional Γ- converges to the parent Griffith-like functional of Francfort and Marigo (1998). Later on, the $\Gamma-$convergence result was demonstrated also in the vectorial case (an exhaustive list of references can be found in Amor et al., 2009).

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[1] The variable s is analogous to the classical damage parameter first introduced by Kachanov (1958). More precisely, Kachanov used the complementary variable $\psi = 1 - s$ called continuity, but the substance does not change.

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It is worth mentioning that the use of supplementary variables, like the aforementioned s, has been pursued in many other physical theories to reproduce sharp interfaces separating regions with qualitatively different properties. For example, in the theory of *liquid crystals* (Virga, 1994), a scalar field representing the degree of orientation was introduced by Ericksen (1991) to regularize certain types of line singularities. More in general, in the class of *phase-field models*, sharp interfaces between heterogenous phases are described as diffused interfaces through an additional field taking, for example in a solidification problem, the 0 values in the liquid and 1 in the solid. The regularization of such a field is obtained through the introduction of gradient terms that induce a smooth transition between the values of the additional field on both sides of the interface. Remarkably, in solid state physics and material science a considerable literature does exist on *phase-field modeling of crack propagation*, where the phases are the sound and disgregated states of the material. Assuming proper equations of evolution, the model of Aranson et al. (2000) is able to capture important phenomena such as crack initiation, crack branching and dynamic fracture instability. A particularly interesting phase-field model of fracture has been recently proposed in Hakim and Karma (2009), which also contains laws of crack-tip motion, stability analysis, generalized Eshelby–Rice integrals and an exhaustive list of relevant references.

In the language of fracture mechanics, the free-discontinuity problem of Francfort and Marigo (1998) represents a *discrete crack model* (de Borst et al., 2004) since the deformation field undergoes a sharp discontinuity at the crack surface. On the other hand, the regularized formulation of Bourdin et al. (2000) can be classified in the category of *smeared crack models* (de Borst et al., 2004) because minimizers of the new functional are characterized by bands, regularized representation of the cracks, having a thickness of the order of l, where the gradients of the displacement field concentrate. Discrete crack models are much more difficult to solve numerically than smeared crack models and, from this point of view, the regularization represents a practical device. The relationship between the smeared model and its parent discrete model is corroborated by the proof of the $\Gamma-$convergence result, even if it should be emphasized that $\Gamma-$convergence considers *global* minima only, even if fracture mechanics typically deals with *local* minima. The regularized model of Bourdin et al. (2000) naturally presents a (pseudo-non-local) gradient term in the damage variable s and introduces the aforementioned internal parameter l, having the dimension of a length, that penalizes extreme strain localization and avoids the ill-posedness of the boundary value problem.

There is, however, another interpretation: the regularized problem is not just an useful approximation, but an independent model *per se* in the class of damage models (Frémond, 2001), with its own physical autonomy being able to reproduce the phenomenon of fracture for those materials, like concrete or geomaterials, usually referred to as *quasi-brittle* (Bažant and Planas, 1998). In such materials microcracks appear first from the sound material in small-width bands usually referred to as the *process zones*, and eventually coalesce to form a main crack, independently of the existence of pre-existing flaws (Bažant and Planas, 1998). In conglomerates like concrete this process is due to the material heterogeneities which force the crack path to follow the contours of the constituent grains. However, experiments have provided evidence that even in glass, the brittle material *par excellence*, crack openings are not atomically sharp as commonly believed (Lawn, 1983), because right at the crack tip a fractal-like boundary layer between sound and fractured material portions has been observed at the microscopic level (Ferretti et al., 2009), which is somehow equivalent to the formation of a process zone. The regularized model of Bourdin et al. (2000) can reproduce such phenomenon and, from this point of view, s is the damage parameter because it interprets the decay of material stiffness at the process zone.

Moreover, the regularized functional embraces material parameters that are lost in the limit free-discontinuity functional. In particular, as discussed at length in Lancioni and Royer-Carfagni (2009), the parameter l disappears in the discrete crack model of Francfort and Marigo (1998), but this represents an information about the material microstructure because it plays the physical role of the material *intrinsic length-scale* that characterizes the width of the process-zone bands. Recall that in conglomerates l equals $2-3$ times the characteristic size of the constituent grains (Bažant and Planas, 1998). Moreover, as highlighted by Charlotte et al. (2006), the free-discontinuity formulation presents the drawback that if local minima are looked for, the load required to open a crack is infinite[2] and the sound elastic state is always locally stable. On the contrary, in the regularized functional of Bourdin et al. (2000) the load required to crack a body is always finite. More precisely, as illustrated in the stability analysis by Bourdin (2007) for the 1-D traction problem, when the fracture opens and the solution evolves from the sound elastic configuration to the cracked condition, an energy barrier has to be jumped over which tends to infinity as l tends to zero. Consequently, the free-discontinuity problem of Francfort and Marigo (1998) cannot recover the Griffith theory without the usual constraint that a pre-existing crack must be assumed *a priori*.

In conclusion, according to this interpretation, the regularized functional is *the model*, while its parent Γ- limit (the free-discontinuity problem) is the *approximation*. In this paper we take this point of view and we study a pseudo-spatial dependent model in the same category of that of Bourdin et al. (2000) and with no need of exhibiting any $\Gamma-$convergence result, but from a much broader viewpoint. In particular, we refer to the Theory of Structured Deformations by Del Piero and Owen (1993) to show that a regularized functional, formally presenting the same structure of that of Bourdin et al.

[2] As a matter of fact, this is not physically inconsistent, because it is in agreement with the most classical tests on glass filaments by Griffith (1921), who experimentally showed that brittle materials without pre-existing flaws are practically unbreakable since their strength approaches the theoretical upper bound of the same order of the elastic modulus.