Available online at www.sciencedirect.com
ScienceDirect
Journal of the Mechanics and Physics of Solids
55 (2007) 1103–1119

JOURNAL OF THE MECHANICS AND PHYSICS OF SOLIDS
www.elsevier.com/locate/jmps

Higher-order macroscopic measures

Gregory J. Rodin*

Institute for Computational Engineering and Sciences, The University of Texas at Austin, Austin, TX 78712, USA

Received 18 August 2006; received in revised form 8 December 2006; accepted 16 December 2006

Dedicated to A.S. Gidaspov on the occasion of his 70th birthday

Abstract

We present generalizations of Hill's classical results concerned with the macroscopic strain and stress measures. Generalizations involve polynomial boundary conditions and polynomial moments of the microscopic fields. It is shown that for higher-order polynomials certain boundary conditions and moments should be excluded from considerations in order to guarantee unique relationships between boundary data and macroscopic measures. Particularly simple relationships are obtained for spherical specimens, for which higher-order macroscopic measures are defined in terms of spherical harmonics. Also it is demonstrated that higher-order macroscopic measures and constitutive equations can be useful in multi-scale analysis of problems formulated in terms of integral equations.

1. Introduction

By definition, the (true) macroscopic strain and stress are the volume averages of the corresponding microscopic fields induced by simple boundary conditions. Typically boundary conditions are associated with constant (apparent) macroscopic strain or stress. Basic relationships between the true and apparent macroscopic measures were formalized by Hill (1963). We state those relationships as follows:

Theorem C (*Compatibility*). *If a solid* Ω *is subjected to the linear displacement*

$$u_i(\boldsymbol{x}) = \varepsilon^0_{ij} x_j, \quad \boldsymbol{x} \in \partial\Omega,$$

*Tel.: +1 512 471 4230.
E-mail address: gjr@ices.utexas.edu.

0022-5096/$ - see front matter © 2007 Published by Elsevier Ltd.
doi:10.1016/j.jmps.2006.12.004

where ε^0_{ij} *is a constant symmetric tensor, then*

$$\bar{\varepsilon}_{ij} = \varepsilon^0_{ij}.$$

Theorem E (*Equilibrium*). *If, in the absence of body forces, a solid* Ω *is subjected to the surface traction*

$$t_i(\boldsymbol{x}) = \sigma^0_{ij} n_i(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega,$$

where σ^0_{ij} *is a constant symmetric tensor, then*

$$\bar{\sigma}_{ij} = \sigma^0_{ij}.$$

Theorem H (*Hill's condition*). *If the conditions of Theorems* C *or* E *hold then*

$$\overline{\sigma_{ij}\varepsilon_{ij}} = \bar{\sigma}_{ij}\bar{\varepsilon}_{ij}. \tag{1}$$

The notation adopted here is standard. All three theorems are elementary to prove using the divergence theorem. A remarkable feature of the theorems is that they hold for any solid, independent of its composition.

Hill's condition (1) holds for other boundary conditions. Those include periodic boundary conditions (Suquet, 1987) and mixed boundary conditions commonly used in material testing (Hazanov, 1998). Furthermore, Hill's condition plays an important role in statistical characterization of representative volume elements (Kroner, 1972; Sab, 1987).

The principal objective of this paper is to generalize Hill's theorems to higher-order macroscopic measures. The generalizations involve polynomial boundary conditions and moments, and they allow one to establish proper definitions for the true macroscopic measures and identify appropriate boundary conditions.

Higher-order macroscopic measures are important for problems in which the microscopic and macroscopic fields cannot be decoupled. That is, when it is impossible to identify the intermediate (mini) length scale associated with the representative volume element. Typically such problems involve strong macroscopic strain and/or stress concentrators (cracks, notches) whose response may be significantly affected by microstructural features (fibers, grains) in their vicinity.

In contrast to their classical counterparts, higher-order macroscopic measures, and consequently the constitutive equations, do depend on the size and shape of the specimen. This severely limits their usefulness for macroscopic models based on partial differential equations. Actually, one may say that the need for representative volume elements is dictated solely by the notion that the macroscopic model is a partial differential equation. However, Hashin (1983) astutely pointed out that it is natural for the macroscopic constitutive equations to be non-local. From this perspective, integral equations offer much better options for developing macroscopic models. Furthermore, in this paper, we demonstrate that higher-order macroscopic measures arise naturally in macroscopic models based on integral equations.

The rest of the paper is organized as follows. In Section 2, we consider the simplest non-classical setting for (thermal) conduction problems. Accordingly we analyze conducting specimens under boundary conditions associated with either linear apparent temperature gradient or linear apparent flux. These problems reveal essential features of higher-order relationships. In Section 3, we consider general monomial and polynomial boundary conditions for conduction and solid mechanics problems. There it is shown that higher-