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On the problem of the stability of a Hamiltonian system with one degree of freedom on the boundaries of regions of parametric resonance☆

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ABSTRACT

A one–degree–of–freedom system that is periodic in time is considered in the vicinity of its equilibrium position in the case of multiple multipliers of the linearized system. It is assumed that the monodromy matrix is reduced to diagonal form and, therefore, the equilibrium is stable in a first approximation. An algorithm for constructing a canonical transformation that brings the system into such a form, in which the terms of high (finite) order are eliminated in the expansion of the Hamiltonian into a time series and the second-order terms are totally absent, is described. The stability and instability conditions are found using Lyapunov's second method and KAM (Kolmogorov–Arnold–Moser) theory in one particular case, in which the stability problem is not solvable for the third- and fourth-order forms in the expansion of the original Hamiltonian into a series.

1. Introduction. Statement of the problem

Suppose a one–degree–of–freedom system, whose motion is described by the canonical equations

$$\frac{dx_1}{dt} = \frac{\partial F}{\partial x_2}, \quad \frac{dx_2}{dt} = -\frac{\partial F}{\partial x_1} \tag{1.1}$$

admits of the equilibrium position $x_1 = x_2 = 0$, in whose vicinity the Hamiltonian can be represented by the converging series

$$F = F(x_1, x_2, t) = \sum_{k=2}^{\infty} F_k(x_1, x_2, t) \tag{1.2}$$

where F_k is a form of order k in x_1 and x_2 with continuous coefficients that are 2π-periodic in t.

Consider linearized system (1.1), which is given by the Hamiltonian F_2. We use $\mathbf{X}(t)$ to denote its fundamental matrix of solutions, which satisfies the condition $\mathbf{X}(0) = \mathbf{E}$, where $\mathbf{E}$ is a second-order unit matrix. We will assume that the characteristic exponents $\pm i\lambda$ of the linearized system are pure imaginary and correspond to the boundaries of regions of parametric resonance. This means that the value of λ is an integer or a half-integer, i.e., there is a first- or second-order resonance, respectively. The roots ρ_j ($j = 1, 2$) of the characteristic equation of the matrix $\mathbf{X}(2\pi)$ (the multiplier) will be multiple: $\rho_1 = \rho_2 = 1$ at the first-order resonance or $\rho_1 = \rho_2 = -1$ at the second-order resonance.

We will assume that the elementary divisors corresponding to them are simple, i.e., we will assume that the matrix $\mathbf{X}(2\pi)$ can be reduced to diagonal form. Then (see, for example, Refs 1 and 2), all the solutions of the linearized system are periodic functions of time with the period τ, which is equal to 2π in the case when $\rho_1 = \rho_2 = 1$ (λ is an integer) and 4π in the case when $\rho_1 = \rho_2 = -1$ (λ is a half-integer), and the equilibrium position $x_1 = x_2 = 0$ of system (1.1) is stable in the first (linear) approximation. In a rigorous solution of the stability problem an analysis of non-linear equations (1.1) is necessary.[1]

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To conduct a non-linear investigation, we preliminarily perform the linear canonical transformation $x_1, x_2 \to y_1, y_2$ using the formulae

$$x_1 = x_{11}(t)y_1 + x_{12}(t)y_2, \quad x_2 = x_{21}(t)y_1 + x_{22}(t)y_2 \tag{1.3}$$

where $x_{ij}(t)$ denotes the elements of the matrix $\mathbf{X}(t)$. The new variables will then correspond to the Hamiltonian $Y(y_1, y_2, t)$, whose expansion into a series in powers of y_1 and y_2 does not contain second-order terms:

$$Y(y_1, y_2, t) = \sum_{k=3}^{\infty} Y_k(y_1, y_2, t) \tag{1.4}$$

Here Y_k is the function F_k from expansion (1.2), in which x_1 and x_2 have been replaced by expressions (1.3). Then, using classical perturbation theory,[3] we construct the canonical, nearly identity replacement of variables $y_1, y_2 \to Q, P$, which is τ-periodic in t and is such that the expansion of the new Hamiltonian $G(Q, P, t)$ into a series in powers of Q and P did not contain time up to terms of the power ℓ inclusive, where the number ℓ can be unrestrictedly large.[4] The set of third-order terms $G_3(Q, P)$ will then be equal to the mean value of the function $Y_3(Q, P, t)$ over the explicitly appearing time.

We will assume that $G_3 \equiv 0$. This requirement always holds at the second-order resonance (when λ is a half-integer). At the first-order resonance (when λ is an integer) it is an additional restriction, which is imposed on expansion (1.2) by the original Hamiltonian and holds, for example, in the simple case when this expansion does not contain forms of odd powers.

Thus, it can be assumed below that the expansion of the function $G(Q, P, t)$ into series has the form

$$G = G_4(Q, P) + G_5(Q, P) + G_6(Q, P) + \tilde{G}(Q, P, t) \tag{1.5}$$

where G_k ($k = 4, 5, 6$) is the form of order k in Q and P, and $\tilde{G}$ is a τ-periodic function of time, whose expansion into series begins from terms that are not lower the seventh order in Q and P.

The system with Hamiltonian (1.5) belongs to the class of systems called[5] strongly non-linear. Constructive methods for analytically representing the solution of such systems were developed in Ref. 5; special attention was focused on establishing the relation between the stability of an equilibrium and the existence of solutions that tend to equilibrium when the time increases or decreases without limit.

The problems of the stability of the equilibrium position $x_1 = x_2 = 0$ of the original system and of the equilibrium $Q = P = 0$ of the transformed system with Hamiltonian (1.5) are equivalent. In many cases the stability and instability conditions can be expressed in terms of the coefficients of G_4 in expansion (1.5). (The assertion that if the form $G_4(Q, P)$ is sign-indefinite and depends on both Q and P, instability occurs (Ref. 6, Theorem 3) is erroneous, as can be seen by considering a system with the Hamiltonian $G = Q^2P^2 + Q^6 + P^6$.)

Suppose $\Phi(\varphi) \equiv G_4(\sin\varphi, \cos\varphi)$. It was previously shown[4,9] using Moser's theorem of invariant curves[7,8] that if the equation $\Phi(\varphi) = 0$ does not have real roots, the equilibrium is stable; if this equation has a real root $\varphi = \varphi_*$ and $d\Phi(\varphi_*)/d\varphi_* < 0$, the equilibrium is unstable. The latter assertion was obtained[4,9] using Lyapunov's second method.

However, cases when the formulated sufficient conditions for stability and instability do not hold are possible. Suppose, for example, $G_4 = QP^3$. Then

$$\Phi = \sin\varphi\cos^3\varphi, \quad d\Phi/d\varphi = \cos^2\varphi(4\cos^2\varphi - 3)$$

The equation $\Phi(\varphi) = 0$ has the real roots $\varphi = \varphi_*$, and either $\sin\varphi_* = 0$ or $\cos\varphi_* = 0$ for them. In the former case the derivative $d\Phi(\varphi_*)/d\varphi_*$ is positive (equal to unity), and in the latter case it is equal to zero.

It was shown in Ref. 4 that by using the linear canonical transformations $Q, P \to q, p$ the fourth-order terms in Hamiltonian (1.5) can be reduced to one of the following nine simple forms, which are not identical to one another:

$$\begin{aligned}
&1)\, q^4 + aq^2p^2 + p^4 (a > -2), \quad 2)\, q^4 + aq^2p^2 + p^4 (a < -2) \\
&3)\, q^4 + aq^2p^2 - p^4 \quad (a - \text{is any real number}), \quad 4)\, q^2(q^2 - p^2) \\
&5)\, q^2(q^2 + p^2), \quad 6)\, q^2p^2, \quad 7)\, q^3p, \quad 8)\, -q^3p, \quad 9)\, q^4
\end{aligned} \tag{1.6}$$

It was also shown that in case 1 the equilibrium under investigation is stable and that in Cases 2, 3, 4 and 7 it is unstable. In Cases 5, 6, 8 and 9 terms above the fourth order in the expansion of Hamiltonian (1.5) must be taken into account to resolve the stability problem.

The results of the investigation only for Case 5 are presented here, and, in accordance with what was stated above, it is assumed that the Hamiltonian has the form

$$H = q^2(q^2 + p^2) + H_5(q, p) + H_6(q, p) + \tilde{H}(q, p, t), \quad H_k = \sum_{\nu+\mu=k} h_{\nu\mu} q^\nu p^\mu \tag{1.7}$$

where the $h_{\nu\mu}$ are constant, and $\tilde{H}$ denotes the set of terms above the sixth order with coefficients that are τ-periodic in time.

2. Simplification of Hamiltonian (1.7) by means of the Birkhoff transformation[10]

We perform the canonical replacement of variables $q, p \to q', p'$, which is defined implicitly by the relations

$$q' = \frac{\partial S}{\partial p'}, \quad p = \frac{\partial S}{\partial q} \tag{2.1}$$