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Dynamics of a Painlevé–Appell system☆

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ABSTRACT

The dynamics of a Painlevé–Appell system consisting of two point masses joined by a weightless rigid rod is studied within two mechanical models, which describe different motion regimes. One of the masses can slide or can be supported at rest on a rough straight line. The boundaries of the region of definition of each of the models are presented, and the transitions between them are analysed for various friction coefficients.

1. Introduction

The Painlevé–Appell problem of the dynamics of a pendulum whose fulcrum can move along a rough horizontal straight line will be investigated. We will assume that the contact with the supporting straight line is a point contact and that it obeys Coulomb's law of dry friction

$$F = -\mu N V/|V|$$

where F is the friction force, which counteracts sliding of the contact point with the velocity V, μ is the friction coefficient, and N is the absolute value of the normal reaction force of the supporting straight line.

Despite the simplicity of the formulation, at certain values of the friction coefficient (which are greater than a certain critical value) this problem exhibits paradoxes associated with the non-existence or non-uniqueness of the solution.

This system was proposed by Painlevé when he studied possible paradoxical solutions.[1] A more detailed investigation was conducted by Appell,[2] who obtained the equations of motion and performed an analysis of the system with small friction coefficients ($\mu < 1$, but the exact value of the critical friction coefficient was not determined). Several particular solutions were investigated in a more general formulation (in the presence of an additional external constraining force h besides the gravitational force).[3] When $h \to 0$, a condition imposed on the friction coefficient, under which sudden stopping can occur as a result of an infinite increase in the friction force, was obtained.

The main results in Refs 2 and 3 are associated with arguments regarding the possibility or impossibility of motion when the contact point has a specific direction of motion. Not only particular solutions, but also the general results of a qualitative analysis of the dynamics, which will enable us below in the example of the classical Painlevé–Appell problem to draw conclusions regarding the behaviour of other systems with bilateral constraints, especially in the vicinity of values of the parameters at which paradoxical solutions are observed and, in addition, enable us to model the experiment and ensure observability of the effects described.

We note that there are many mechanical systems with Coulomb friction at a contact point and unilateral or bilateral constraints in which such paradoxes are encountered (for example, a body supported on a rough plane by one contact point,[4] a brake shoe,[5] a ladder resting on a horizontal floor and a vertical wall,[2,6] a Painlevé–Klein system[7] etc.). Despite the long history of the investigation of the systems indicated (beginning with Painlevé's classical work[1] and the subsequent discussion of the well-known mechanicians at the beginning of the twentieth century), an increase in interest in this subject was observed recently, especially in the investigation of solutions in the vicinity of paradoxical regions (for examples of modern analytical approaches using variational and asymptotic methods, see, e.g., monographs[8,9] and the references therein).

Apart from the development of mathematical methods of investigation, there are two more principal reasons for the resumption of investigations of classical problems. First, computer simulation and visualization methods, which are needed, in particular, for the

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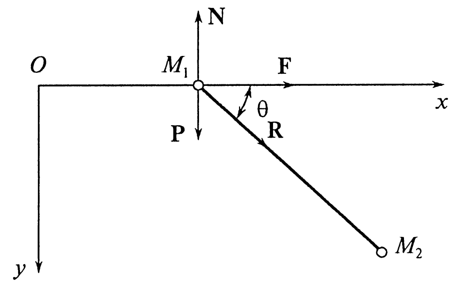

Fig. 1.

construction and analysis of phase portraits and regions of possible motion (RPMs), are undergoing rapid development. Analytical methods combined with the use of modern computer methods along with the development of bifurcation theory enable us to reveal qualitative features of the dynamics for various values of the parameters of the system.[4] Second, the experimental facilities and methods for the high-speed recording of natural experiments and the data processing for observing different kinds of dynamic effects have been improved significantly. The main difficulty in setting up an experiment lies in establishing the initial conditions and parameters of the system that bring the system to a paradoxical solution (generally, the value of the critical friction coefficient for the problems just described is considerably greater than the values observed in nature). For this reason, new, more complex mechanical systems (for example, an inverted pendulum on a slider (IPOS),[10] dynamic walkers[11] etc.), whose theoretical analysis has revealed paradoxical solutions at values of the system parameters that are obtained in a natural experiment, are being developed. We also note that the opposite situation, in which so-called non-intuitive (non-obvious) motion is observed under easily implemenable initial conditions, appears for several systems (with rolling).[12–14]

Thus, to set up an experiment, it is first necessary to analyse the dynamics of the system as a whole and to model the behaviour of the system to obtain a more complete description of the theoretically predicted effects. As a rule, particular solutions alone do not suffice since theory and experiment cannot always provide the necessary quantitative agreement under the chosen initial conditions. Nevertheless, in a certain range of initial data it often suffices to obtain a qualitative agreement between experiment and theory.

Below, we will use modern computer capabilities and methods of qualitative analysis (that were developed for a system with a unilateral constraint[4]) to investigate the dynamics of a Painlevé–Appell system and the special features of the solutions for friction coefficients that are smaller and greater than the critical value within two mechanical models: a pendulum and a sliding rod. The use of an illustrative method to construct the phase portraits has enabled us to consider the case when the friction coefficient is greater than the critical value even on the boundaries of the regions of possible motion when the motion mode changes.

2. Equations of motion and boundaries of regions of possible motion

A Painlevé–Appell system (for a schematic representation, see Fig. 1) is a planar pendulum whose fulcrum M_1 can slide along the rough straight line Ox with the friction coefficient μ. The pendulum is a rigid weightless rod of length l with two point masses M_1 and M_2 of identical mass m attached to its ends.

We introduce the following notation: θ is the angle xM_1M_2, x_1 is the abscissa of the point M_1, and (x_2, y_2) denotes the coordinates of the point M_2. The following forces act on the point M_1: the weight $P = mg$, the reaction R of the rod, and the normal component N and the horizontal component F of the reaction of the straight line Ox. The point M_2 is acted upon by the weight $P_2 = mg$ and the reaction R of the rod.

If the force of the horizontal reaction F along the directrix Ox at the contact point does not exceed a certain limiting value (which depends on the friction coefficient), the system is set in motion in which the contact point does not slide, and the rod performs oscillation that can be described by the equations of a mathematical pendulum. If the reaction F has the limiting value, the system slides at the contact point and can then perform oscillatory motions. Therefore (taking into account that the constraint is retaining at the point M_1), we will consider the motion of the system within two mechanical models: a) a pendulum ($\dot{x}_1 = 0$), b) a sliding rod ($\dot{x}_1 \neq 0$).

We will write all the equations below in dimensionless form. For this purpose, as the units of measure of length, time and force we choose the quantities l, $\sqrt{l/g}$ and mg.

Pendulum. In the pendulum model the contact point M_1 does not slide ($\dot{x}_1 = 0$). In this case the configuration of the system is determined by the single generalized coordinate θ, for which the equation of motion (the equation of a mathematical pendulum) has the form

$$\ddot{\theta} = \cos\theta \tag{2.1}$$

The region in which this model is applicable corresponds to the no-slip condition of point M_1, and the total reaction $\mathbf{F} + \mathbf{N}$ (see Fig. 1) from the straight line Ox should lie within the cone of friction:

$$\mu|N| > |F| \tag{2.2}$$

To determine the regions corresponding to inequality (2.2), we write the equations of motion of points M_1 and M_2 in Cartesian coordinates:

$$\begin{aligned} \ddot{x}_1 &= F + R\cos\theta, \quad \ddot{y}_1 = 1 + N + R\sin\theta \\ \ddot{x}_2 &= -R\cos\theta, \quad \ddot{y}_2 = 1 - R\sin\theta \end{aligned} \tag{2.3}$$