Journal of Applied Mathematics and Mechanics 80 (2016) 16–23

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Optimal conditions with chattering in the inverted two-link pendulum control problem☆

M.N. Ronzhina

M. V. Lomonosov Moscow State University, Moscow, Russia

ARTICLE INFO

Article history:
Received 26 March 2015
Available online 16 June 2016

ABSTRACT

The plane motion of a two-link inverted mathematical pendulum, attached by a hinge to a moving trolley, is studied. The pendulum is controlled by a bounded force applied to the trolley. The problem of the minimization of the mean square deviation of the pendulum from an unstable equilibrium position is considered. Pontryagin's maximum principle is used. An optimal feedback control, containing special second order trajectories and trajectories with chattering is constructed for a linearized model. It is proved that, before emerging onto a special manifold, the optimal trajectories experience a chattering after a finite period of time and then reach the unstable equilibrium after an infinite time by a specific mode. The global optimality of the solution constructed is proved.

Problems of the local and global stabilization of an unstable equilibrium state of an inverted two-link pendulum, when its motion is controlled by a bounded torque applied between the base and the lower link of the pendulum or to the hinge between the links, have been considered in the case of fixed suspension point [1,2] and in the case of a pendulum on a fixed base.[3] The phase trajectories when there is no control have been constructed for the system of a one-link pendulum on a wheel and the control that ensures the global stabilization of the upper equilibrium position, and the time-optimal feedback control has been constructed.[3]

As before [3], the problem of controlling an inverted two-link pendulum on a moving base is considered below. The difference lies in the fact that a torque applied to the a force acting on the moving base is taken as the control rather than lower link. Moreover, a control is chosen that does not simply stabilize the pendulum in the upper equilibrium position but minimizes the mean square deviation of the pendulum from this position.

1. Equations of motion

We will consider the problem of controlling a plane, two-link pendulum attached by a hinge to a trolley moving along a line (along the x axis, Fig. 1). The links of the pendulum OD and DE are absolutely rigid rods and they are connected by a hinge at the point D. The friction in the hinges is not taken into account. A horizontal control force $|u| \le 1$ acts on the trolley.

Suppose α_i is the angle of decline of a link from the vertical measured counterclockwise, the subscript i =1corresponds to the lower link of length l and the subscript i = 2 to the upper link, M is the mass of the trolley, m_i, r_i and I_i are the mass of a link, the distance from the lower end to the centre of mass of a link and the moment of inertia of the link with respect to its centre of mass, and s is the position of the trolley on the X axis.

The kinetic energy of the system is the sum of the three components:

$$T = T_0 + T_1 + T_2; \quad T_0 = \frac{1}{2}M\upsilon_0^2, \quad T_i = \frac{m_i\upsilon_i^2}{2} + \frac{1}{2}I_i\dot{\alpha}_i^2$$

where T_0 and T_i are the kinetic energies of the trolley and of a link.

☆ *Prikl. Mat. Mekh.*, Vol. 80, No. 1, pp. 24–33, 2016.
E-mail address: maryaronzhina@gmail.com

http://dx.doi.org/10.1016/j.jappmathmech.2016.05.004
0021-8928/

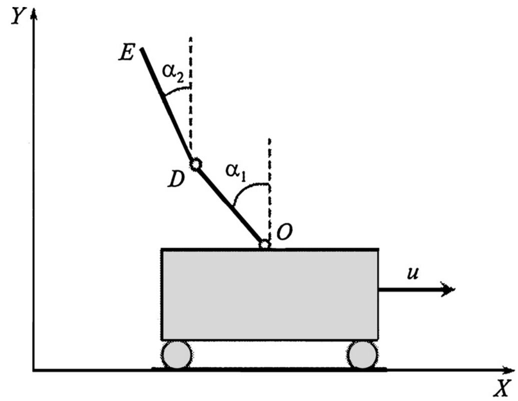

Figure 1.

The expressions for the squares of the velocities of the trolley (the subscript 0) and the links (the subscripts 1 and 2) have the form

$$\upsilon_0^2 = \dot{s}^2$$
$$\upsilon_1^2 = \dot{s}^2 - 2\dot{s}\dot{\alpha}_1 r_1 \cos\alpha_1 + r_1^2\dot{\alpha}_1^2$$
$$\upsilon_2^2 = \dot{s}^2 + l^2\dot{\alpha}_1^2 + r_2^2\dot{\alpha}_2^2 - 2\dot{s}(\dot{\alpha}_1 l\cos\alpha_1 + \alpha_2 r_2\cos\alpha_2)$$
$$+ 2lr_2\dot{\alpha}_1\dot{\alpha}_2\cos(\alpha_1 - \alpha_2)$$

Similarly, the force function of the system has the form

$$U = U_0 + U_1 + U_2; \quad U_0 = 0, \quad U_i = -m_i g r_i \cos\alpha_i$$

where U_0 and U_i are the force functions of the trolley and of a link.

We therefore obtain

$$T = a_{11}\dot{s}^2 + a_{12}\dot{s}\dot{\alpha}_1\cos\alpha_1 + a_{13}\dot{s}\dot{\alpha}_2\cos\alpha_2 + a_{22}\dot{\alpha}_2^2$$
$$+ a_{23}\dot{\alpha}_1\dot{\alpha}_2\cos(\alpha_1 - \alpha_2) + a_{33}\dot{\alpha}_2^2$$
$$U = -b_1\cos\alpha_1 - b_2\cos\alpha_2 \qquad (1.1)$$

where

$$a_{11} = \frac{1}{2}(M + m_1 + m_2), \quad a_{12} = -m_1 r_1 - m_2 l, \quad a_{13} = -m_2 r_2$$
$$a_{22} = \frac{1}{2}(m_1 r_1^2 + m_2 l^2 + I_1), \quad a_{23} = m_2 l r_2, \quad a_{33} = \frac{1}{2}(m_2 r_2^2 + I_2)$$
$$b_1 = m_1 g r_1 + m_2 g l, \quad b_2 = m_2 g r_2 \qquad (1.2)$$

We write the equations of motion of the pendulum with the control force u (henceforth $i = 1, 2$ everywhere) as

$$a_{11}\ddot{s} + a_{12}\ddot{\alpha}_1\cos\alpha_1 + a_{13}\ddot{\alpha}_2\cos\alpha_2 - a_{12}\dot{\alpha}_1^2\sin\alpha_1 - a_{13}\dot{\alpha}_2^2\sin\alpha_2 = u$$
$$a_{1\,i+1}\ddot{s}\cos\alpha_i + a_{i+1\;i+1}\ddot{\alpha}_i + a_{23}\ddot{\alpha}_{3-i}\cos(\alpha_1 - \alpha_2)$$
$$+ (-1)^{i+1}a_{23}\dot{\alpha}_{3-i}^2\sin(\alpha_1 - \alpha_2) - b_i\sin\alpha_i = 0 \qquad (1.3)$$

When $u = 0$, the system of equations (1.3) has the trivial solution

$$s = \dot{s} \equiv 0 \qquad (1.4)$$
$$\alpha_i = \dot{\alpha}_i \equiv 0 \qquad (1.5)$$

corresponding to the unstable equilibrium position.

The problem of finding the feedback control that minimizes the mean square deviation of the pendulum from the upper unstable equilibrium position is set:

$$\int_0^\infty (\alpha_1^2(t) + \alpha_2^2(t))dt \to \min \qquad (1.6)$$