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A qualitative analysis of sets of trajectories of mechanical systems☆

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ABSTRACT

The evolution of geometric measures (volume, surface area) of sets of attainability of linear controlled mechanical systems with constant parameters is studied. Lyapunov's direct method, the comparison method, and theory of mixed volumes are used. Based on the general comparison theorem, estimates are obtained for the solutions of differential equations with a generalized Hukuhara derivative that describe the evolution of regions of attainability. For linear controlled systems with one degree of freedom, the maximum boundedness conditions are obtained for the area of the set of attainability. Examples of the application of the obtained results are given.

1. Introduction

The problem of approximating the sets of attainability (SAs) of non-linear controlled systems is closely related to the theory of differential equations with multivalued right-hand parts,[1,2] in particular to differential equations with a Hukuhara derivative.[3] We will examine the non-linear controlled system

$$\frac{dx}{dt} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}), \quad t \in \mathbb{R}, \quad \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{u} \in \mathbb{R}^m, \quad \mathbf{f}: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \tag{1.1}$$

Vector $\mathbf{u}$ can be interpreted either as the control (in the case of a controlled system) or as a disturbance vector (in the case of a system under conditions of incomplete information). We will assume that upon vector $\mathbf{u}$ and upon the set u_0 of initial data are imposed the constraints

$$\mathbf{u} \in \mathbf{U}(t, \mathbf{x}), \quad \mathbf{x}(t_0) \in u_0 \in \mathrm{conv}\mathbb{R}^n \tag{1.2}$$

where $\mathrm{U} : \mathbb{R} \times \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is a map whose region of values consists of compact sets in $\mathbb{R}^m$, and $\mathbb{R}^n$ is the metric space of non-empty convex compacts of space $\mathbb{R}^n$ with a Hausdorff metric. In correspondence to the controlled system (1.1) with constraints (1.2) is placed the differential inclusion

$$\dot{\mathbf{x}} \in \mathbf{X}(t, \mathbf{x}), \quad \mathbf{x}(t_0) \in u_0 \tag{1.3}$$

where $\mathbf{X}(t, \mathbf{x}) = \mathbf{f}(t, \mathbf{x}, \mathbf{U}(t, \mathbf{x}))$ is the set of all possible values of the right-hand parts of system (1.1).

With differential inclusion (1.3), a differential equation with a Hukuhara derivative

$$D_H u(t) = \mathrm{co}\mathbf{X}(t, u(t)), \quad u(t_0) = u_0 \in \mathrm{conv}\mathbb{R}^n \tag{1.4}$$

is associated, where co$\mathbf{X}$ is the convex hull of set $\mathbf{X}$.

Let $u(t\,; t_0, u_0)$ denote the solution of the Cauchy problem for Eq. (1.4). It has been demonstrated[1] that the set $u(t\,; t_0, u_0)$ covers the region of attainability $D(t\,; t_0, u_0)$ of controlled system (1.1) with constraints (1.2), i.e.,

$$D(t; t_0, u_0) \subset u(t; t_0, u_0), \quad t \geq t_0 \tag{1.5}$$

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Differential equations with a Hukuhara derivative have been the subject of a considerable number of studies; in particular, Lyapunov's direct method and its generalization – the comparison method – have been developed[3] for investigating the various dynamic properties of the indicated class of differential equations. However, the question of the choice of specific Lyapunov functions is not addressed in these studies, or it is pared down to the consideration of trivial one-dimensional examples.

Below, new qualitative (geometric) methods are proposed for investigating the sets of trajectories of non-linear controlled mechanical systems. These methods are based on the classic ideas of Lyapunov, on ideas of the comparison method combined with theory of mixed volumes dating back to studies by Minkowski and Aleksandrov, and also a certain slight modification of the concept of the Hukuhara derivative.

The principal general result is a description of the evolution of certain geometric measures of sets of trajectories of controlled mechanical systems (in the particular case, these measures are the volume or surface area of the sets of trajectories). The obtained results make it possible to assess these measures during evolution of the system. The general approaches presented are used to investigate the geometry of SAs of linear controlled mechanical systems with one degree of freedom. Note that the obtained results also make it possible indirectly to assess the accuracy of various methods of approximation of SAs of controlled systems. The proposed approaches lead to a qualitative analysis of the dynamics of fairly complex control systems, and also systems under conditions of incomplete information.

2. Secondary results

Let $\mathrm{conv}\mathbb{R}^n$ be the metric space of convex compacts of $\mathbb{R}^n$ with a Hausdorff metric; the operations of addition (Minkowski) and multiplication by a non-negative scalar are determined within this space. If $\mathbf{A} \in L(\mathbb{R}^n)$, where $L(\mathbb{R}^n)$ is the linear operator space in $\mathbb{R}^n$, then the action of operator $\mathbf{A}$ naturally propagates to space $\mathbb{R}^n$:

$$\mathbf{A}u = \{\mathbf{A}x : x \in u\} \in \mathrm{conv}\mathbb{R}^n, \quad u \in \mathrm{conv}\mathbb{R}^n$$

Let $u, \upsilon \in \mathrm{conv}\mathbb{R}^n$. Then, if the element $w \in \mathrm{conv}\mathbb{R}^n$ exists such that $u = w + \upsilon$, the element $u = w - \upsilon$ is termed the Hukuhara difference of elements u and υ; it does not always exist.

The concept of the Hukuhara difference makes it possible to define the concept of the Hukuhara derivative for the map

$$F : (\alpha, \beta) \to \mathrm{conv}\mathbb{R}^n, \quad (\alpha, \beta) \subset \mathbb{R} \tag{2.1}$$

Definition 2.1. Map (2.1) is termed differential at the point $t_0 \in (\alpha, \beta)$ if the element $D_H F(t_0) \in \mathrm{conv}\mathbb{R}^n$ exists such that the limits

$$\lim_{\varrho \to 0+} \frac{F(t_0 + \varrho) - F(t_0)}{\varrho}, \quad \lim_{\varrho \to 0+} \frac{F(t_0) - F(t_0 - \varrho)}{\varrho} \tag{2.2}$$

exist and are equal to $D_H F(t_0)$ – the Hukuhara derivative at point t_0. Differentiability on open, half-open, and closed intervals is determined in the standard way.

The map $F(t)$, differentiated on $[a, b] \subset \mathbb{R}$, is restored in terms of its derivative using the Aumann integral[3]

$$F(t) = F(a) + \int_a^t D_H F(s) ds, \quad t \in [a, b]$$

A necessary condition of map differentiability is a non-decrease in the function $\mathrm{diam}F(t)$.

We will give some results, necessary for further exposition, concerning the geometry of convex bodies, following Aleksandrov.[4]

Let $u_i \in \mathrm{conv}\mathbb{R}^n$ ($i = 1, \ldots, n$), $V[u_1, \ldots, u_n]$ be a mixed volume of convex compacts u_i, and $V[u] = V[u, \ldots, u]$ be the volume of the body $u \in \mathrm{conv}\mathbb{R}^n$.

The functional $V[u_1, \ldots, u_n]$ is additive and positive homogeneous for each argument, invariant relative to the rearrangement of arguments, and also continuous in the collection of its arguments relative to the Hausdorff metric and monotonic in inclusion.

As the Hukuhara difference of two elements of space $\mathrm{conv}\mathbb{R}^n$, like the derivative of the map $(\alpha, \beta) \to \mathrm{conv}\mathbb{R}^n$, is not always defined, it is necessary to embed the space $\mathrm{conv}\mathbb{R}^n$ in some Banach space so that the (Hukuhara) difference of any two elements of the space is always defined as an element of this broader space, and the concept of the derivative of the map will be applicable to a broader class of maps. Such embedding was done back in 1937 by Aleksandrov.[4] Similar results in this regard were also given later on.[5–7] Before giving the corresponding construction, we will note the following. The space $\mathrm{conv}\mathbb{R}^n$ is embedded isometrically and isomorphically as a wedge in the space $C(S^{n-1})$ – continuous functions on a unit sphere S^{n-1}. Such embedding is done by comparing with each element of $u \in \mathrm{conv}\mathbb{R}^n$ its support function. Therefore, below, the elements of the space $\mathrm{conv}\mathbb{R}^n$ will be identified with their support functions, and no special mention of this will be made.

We will describe the embedding of space $\mathrm{conv}\mathbb{R}^n$ in a certain linear normalized space A^n such that in this space for any two elements the operation of difference of the two elements is feasible.

Let us examine the set $\mathrm{conv}\mathbb{R}^n \times \mathrm{conv}\mathbb{R}^n$ and introduce on it the binary ratio ρ

$$(u, \upsilon)\rho(w, z) \equiv (u + z = \upsilon + w)$$

Let

$$\mathrm{A}^n = \mathrm{conv}\mathbb{R}^n \times \mathrm{conv}\mathbb{R}^n / \rho$$

In space A^n, the operations of addition and multiplication by the scalar $\lambda \in \mathbb{R}$ are introduced. If

$$[(u, \upsilon)] \in \mathrm{A}^n, \quad [(w, z)] \in \mathrm{A}^n$$