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# Homothetic radial solutions of the Newtonian general spatial $(N+1)$-body problem☆

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ABSTRACT

The equations of motion are derived, and the existence of symmetrical radial solutions of the general spatial $(N+1)$-body problem, in which $N$ bodies of identical mass $m$ are found at the initial instant of time at the vertices of polyhedrons known as Platonic bodies ($N$ = 4, 6, 8, 12, 20) and the $(N+1)$-th body of mass $M$ is found at the geometrical centre of the polyhedron, is proved rigorously, analytically and numerically. It is assumed that all the bodies are attracted according to Newton's law and that the initial velocities of the bodies with mass $m$ are directed along a radius vector. The solutions found are homothetically expanding (contracting) central configurations. A geometrical image of the solutions is presented for a hexahedron, and their evolution is described for all the Platonic bodies. A distinguishing characteristic of the solutions is the value of the rate of expansion (contraction) of a configuration, which depends on the attraction law, the number of vertices in the polyhedron and its configuration.


According to the statements of several authors,[1–3] the formulation of the concept of a central configuration belongs to Laplace. However, Dziobek[4] was apparently one of the first who began to use the terms "central figures" and "configurations". After analysing, generally speaking, the few publications on the investigation of central configurations up until 1940, Wintner[2] formulated several theorems, presented rigorous analytical proofs of the conditions for the existence of central configurations and developed a very simple classification after identifying homographic and homothetic central configurations and relative equilibrium positions. Based on the work of Hoppe[5] and Lehmann-Filhés,[1] Wintner[2] noted the possibility of the existence of three-dimensional central configurations in the form of regular polyhedrons similar to the five well-known Platonic bodies. It noteworthy that Wintner did not mention the hexahedron (cube, $N$ = 8) when he listed the aforementioned bodies.

There is a fairly extensive bibliography regarding central configurations (even if it is restricted only to publications regarding the conditions for their existence and stability); however, in the overwhelming majority of cases, they generally dealt only with *planar central configurations*. The focus of our investigation is spatial central configurations.

Investigators turned their attention to the analysis of *spatial central configurations* in the general $(N+1)$-body problem only at the end of the past century.[6–16] Grebenikov et al.[9] called spatial solutions in the form of Platonic bodies *radial solutions* and noted that such solutions exist only under the condition that the initial velocities of all the bodies located at the vertices of the aforementioned regular polyhedrons are directed along the corresponding radius vector away from or toward the geometrical centre of the figure. In the paper just cited, differential equations of motion of the bodies were presented (without deriving them), and the possibility of solving them was noted.

The derivation of differential equations of motion of bodies of mass $m$ in the Newtonian general spatial $(N+1)$-body problem within regular polyhedrons is presented below, their general integral and particular solutions are obtained, and the nature of the motion is discussed.

## 1. Statement of the problem and equations of motion

We will consider the general $(N+1)$-body problem, in which at the initial instant of time $N$ bodies of identical mass $m$ are found at the vertices of the polyhedrons known as Platonic bodies with the number of vertices $N$ = 4, 6, 8, 12, 20) and the $(N+1)$-th body of mass $M$ is

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found at the geometrical centre of the respective polyhedron. We will assume that all the bodies are attracted according to Newton's law and that the bodies of mass $m$ have an initial velocity which is equal in absolute value and directed along the radius vector toward or away from the centre of the polyhedron.

We will use the general form of the differential equations of motion of a system of mutually attracted bodies (points masses) $m_i$ in the relative spherical system of coordinates ($r$, $\lambda$, $\varphi$) with the origin of coordinates at the centre of the body of mass $M$ (Ref. 17)

$$\frac{d^2r_i}{dt^2} - r_i\dot{\varphi}_i^2 - r_i\dot{\lambda}_i^2\cos^2\varphi_i = \frac{\partial U}{\partial r_i}$$
$$\frac{d}{dt}(r_i^2\dot{\varphi}_i) + \frac{1}{2}r_i^2\dot{\lambda}_i\sin 2\varphi_i = \frac{\partial U}{\partial \varphi_i}$$
$$\frac{d}{dt}(r_i^2\dot{\lambda}_i\cos^2\varphi_i) = \frac{\partial U}{\partial \lambda_i} \tag{1.1}$$

where the force function has the well-known form

$$U(r_i, \lambda_i, \varphi_i) = \frac{f(M+m_i)}{r_i} + f\sum_{j=1; j\neq i}^{N} m_j\left(\frac{1}{\Delta_{ij}} - \frac{r_i\cos\gamma_{ij}}{r_j^2}\right)$$

$$\Delta_{ij} = \sqrt{r_i^2 + r_j^2 - 2r_ir_j\cos\gamma_{ij}}, \quad \cos\gamma_{ij} = \sin\varphi_i\sin\varphi_j + \cos\varphi_i\cos\varphi_j\cos(\lambda_i - \lambda_j)$$

$f$ is the gravitational constant (we set it equal to unity), $m_i = m$ ($i = 1,\ldots, N$), and $\gamma_{ij}$ is the angle between the radius vectors $r_i$ and $r_j$ of bodies with masses equal to $m$.

Substituting the partial derivatives of the force function into system (1.1) and taking into account the initial conditions

$$r_i(0) = a_0,\ \dot{r}_i(0) = b_0, \quad \lambda_i = \lambda_{i0} = \text{const}, \quad \varphi_i = \varphi_{i0} = \text{const}$$

we will seek the radial solutions $r_i = r_i(t)$. By virtue of the obvious equalities

$$\dot{\lambda}_i = \ddot{\lambda}_i = 0, \quad \dot{\varphi}_i = \ddot{\varphi}_i = 0$$

system (1.1) reduces to the differential equation

$$\frac{d^2r_i}{dt^2} = -\frac{M+m}{r_i^2} + m\sum\left(\frac{r_j\cos\gamma_{ij} - r_i}{\Delta_{ij}^3} - \frac{\cos\gamma_{ij}}{r_j^2}\right) \tag{1.2}$$

which, after some transformations and introduction of the notation $r_j/r_i = p$ = const in accordance with Laplace's theorem, can be written in the form

$$r_i^2\frac{d^2r_i}{dt^2} = -M + A(m, N, p) \tag{1.3}$$

$$A(m, N, p) = -m + m\sum\left(\frac{p\cos\gamma_{ij} - 1}{(1 + p^2 - 2p\cos\gamma_{ij})^{3/2}} - \frac{\cos\gamma_{ij}}{p^2}\right) \tag{1.4}$$

In the case of regular polyhedrons and identical initial velocities of the bodies along the radius vectors, the equality $p = 1$ holds, since (owing to the geometrical and gravitational symmetries of the configurations of the bodies) the rate of variation of the radius vectors, which are equal at the initial instant of time, is identical.

The following value was previously presented[9] without derivation for central configurations in the form of a cube:

$$A(m, 8, 1)/m = \left(3\sqrt{6} + 3\sqrt{3} + \sqrt{2}\right)/4\sqrt{2}$$

This result is obtained from expression (1.4) by a simple calculation using the scheme presented in Fig. 1 and the corresponding equalities

$$\cos\gamma_{1\alpha} = 1/3, \quad \cos\gamma_{1\beta} = -1/3, \quad \cos\gamma_{17} = -1; \quad \alpha = 2, 4, 5; \quad \beta = 3, 6, 8$$

The results of the calculations of $A(m, N, 1)/m$ are presented below for the numbers of vertices $N = 4, 6, 12, 20$.

| Polyhedron | $N$ | $A(m, N, 1)/m$ |
|---|---|---|
| tetrahedron | 4 | $\left(8 + 3\sqrt{3}\right)/2$ |
| octahedron | 6 | $\left(1 + 4\sqrt{2}\right)/4$ |
| icosahedron | 12 | $\left(5\left(\sqrt{5+\sqrt{5}} + \sqrt{5-\sqrt{5}}\right) + \sqrt{2}\right)/4\sqrt{2}$ |
| dodecahedron | 20 | $\left\{\sqrt{6}\left(\sqrt{5} + 1 + \sqrt{2}/2\right) + \sqrt{2}/2 + 3\left(1/\sqrt{9\sqrt{5}-7} + 1/\sqrt{9\sqrt{5}+7}\right) + 3\left(1/\sqrt{4\sqrt{5}-1} + 1/\sqrt{4\sqrt{5}+1} + 1/\sqrt{4\sqrt{5}+2} + 1/\sqrt{10\sqrt{5}-1}\right)\right\}/2\sqrt{2}$ |