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The existence and stability of generalized planar central configurations of the trapezoidal type with a non-spherical body at their centre☆

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ABSTRACT

The existence of trapezoidal planar central configurations (CCs) without a central body and with a spherical central body (the classic case), and also with a central body in the form of an ellipsoid, either homogeneous or inhomogeneous but consisting of ellipsoidal layers of constant density (generalized cases), is demonstrated. It is shown that such CCs exist in a heliocentric system of coordinates with mutually perpendicular diagonals of the trapezia, at the vertices of which the remaining bodies are positioned. The stability of the generalized trapezoidal planar CCs is investigated. It is established that most versions of such CCs do not possess Lyapunov stability.

The study of central configurations (CCs) is of interest not only for celestial mechanics but also for many areas of mathematical analysis, differential equations, analytical mechanics, stellar dynamics, and the dynamics of space flight.

The existence of planar CCs in problems of four or five bodies was demonstrated in the late nineteenth/early twentieth century[1–6] and in the 1930s.[7–9] An analysis of previous studies made it possible to formulate strict definitions and theorems of existence for CCs.[10] Among the studies of the stability of planar CCs, we will single out systematic investigations into problems of the stability of various classic planar CCs.[11] The problem of the stability of a trapezoidal CC has not been entirely settled because of the cumbersome nature of the characteristic determinant. In the past two decades, many CCs have been examined within the framework of the problem of N bodies.[12–15] The influence of the dominant mass on the behaviour of bodies in a CC has been noted and analysed.[16,17] In all the studies mentioned, it was assumed that the bodies entering the CC are spheres.

The existence of generalized planar CCs in certain versions of the problem of five bodies with a non-spherical central body has been demonstrated: square CCs with a triaxial ellipsoid;[18,19] a delta-shaped CC;[20,21] a trapezoidal CC with a spheroid.[20,22]

To supplement the given results, the existence of trapezoidal planar CCs both in classic and in generalized versions of the problem with the entire figure spectrum of the central bodies is demonstrated below, and their Lyapunov stability is also investigated (for the first time for generalized versions of CCs).

1. Problem statement. The necessary and sufficient conditions for the existence of a trapezoidal planar central configuration

Let us examine the motion of bodies of identical mass m positioned at the vertices of an isosceles trapezium with mutually perpendicular diagonals and with a central body of mass M, and attracted according to Newton's law[20,22] (Fig. 1). We have a planar central configuration (CC) – a classic configuration if the central body is a sphere, and a generalized configuration if the body is non-spherical.

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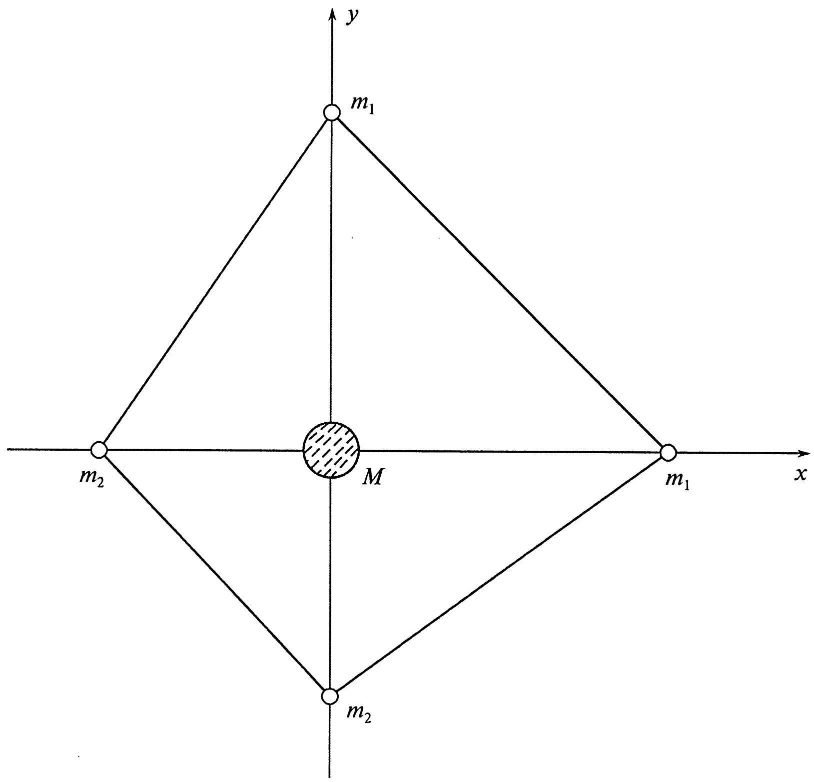

Fig. 1.

The equations of motion of four bodies with equal masses in pairs m_i (i = 1, 2) in a relative heliocentric system of coordinates rotating at constant angular velocity ω about a central body of mass M have the form[20,23]

$$\ddot{x}_i - 2\omega\dot{y}_i = (\omega^2 - \alpha_i)x_i + f\sum\beta_j(x_j, x_i), \quad \ddot{y}_i + 2\omega\dot{x}_i = (\omega^2 - \alpha_i)y_i + f\sum\beta_j(y_j, y_i)$$
$$\ddot{z}_i = -\alpha_i z_i + f\sum\beta_j(z_j, z_i) \tag{1.1}$$

$$\alpha_i = f(M + m_i)\frac{1}{r_i^3}, \quad \beta_j(s_j, s_i) = m_i\left(\frac{s_j - s_i}{\Delta_{ij}^3} - \frac{s_j}{r_j^3}\right)$$

$$r_i = \sqrt{x_i^2 + y_i^2 + z_i^2}, \quad \Delta_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}, \quad i, j = 1, 2$$

Here and below, unless stated otherwise, summation with respect to j is conducted from 1 to 2 with $j \neq i$, and f is the gravitation constant, which without loss of generality we assume is equal to unity.

Let us examine the stationary solutions of system (1.1)

$$x_i = \bar{x}_i = \text{const}, \quad y_i = \bar{y}_i = \text{const}, \quad z_i = \bar{z}_i = 0 \tag{1.2}$$

that define planar CCs. The necessary and sufficient conditions for the existence of CCs will be that the right-hand parts of Eqs (1.1) are equal to zero.

Let us examine two cases of generalizationion of the problem of the shape of the central body: a spheroid or a triaxial ellipsoid. In these cases, in the system of equations (1.1), the terms containing factor M should be replaced with other terms taking into account the shape of the central body.

For a spheroid we have the expansion of the potential (the force function) in terms of the Legendre polynomials[23] in the form

$$U(x, y, z) = \frac{Mm_i}{r}\sum_{n=0}^{\infty}\frac{3(-1)^n\tilde{\upsilon}^{2n}}{(2n+1)(2n+3)}\left(\frac{c}{r}\right)^{2n}P_{2n}(\tilde{z}) = \frac{Mm_i}{r}\left[1 - \frac{1}{5}\xi^2P_2(\tilde{z}) + \frac{3}{35}\xi^4P_4(\tilde{z}) - \ldots\right]$$
$$P_2(\tilde{z}) = \frac{1}{2}(-1 + 3\tilde{z}^2), \quad P_4(\tilde{z}) = \frac{1}{8}(35\tilde{z}^4 - 30\tilde{z}^2 + 3)$$
$$r = \sqrt{x^2 + y^2 + z^2}, \quad \tilde{z} = \frac{z}{r}, \xi = \frac{\tilde{\upsilon}c}{r}, \quad (\tilde{\nu}c)^2 = a^2 - c^2, \quad a = b > c \tag{1.3}$$

where m_i is the mass of the body positioned at a vertex (i = 1, 2).