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A smoothing filter based on an analogue of a Kalman filter for a guaranteed estimation of the state of dynamical systems☆

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ABSTRACT

The equations of a smoothing filter are obtained based on a filter for the guaranteed estimation of the state of dynamical systems, similar to the classical Kalman filter. Particular attention is paid to the link between the probabilistic approach to the processing of inexact measurements of the phase vector of dynamical systems with uncertainties and the method of ellipsoids. The procedure for applying the relations obtained is demonstrated using a simple mechanical system as an example.

1. Introduction

It is well known that the different methods of describing uncertain quantities in dynamical systems are of great interest both from the point of view of theoretical investigations and in practical applications. The most common approach is the probabilistic (stochastic) method. In this method, a certain probability distribution with a specified density is matched with each uncertain vector and the theory of random processes serves as the mathematical apparatus. We stress that the probabilistic approach requires a knowledge of the statistical characteristics of the initial uncertain factors, that is far from always accessible in practice.

The guaranteed (minimax) approach operates with the sets in which the uncertain vectors lie. It is assumed here that the unknown noises are localized in known sets, but in other respects are arbitrary. We will now consider it in greater detail.

Henceforth, all the time functions are such that the solutions of the differential equations in which these functions are used exist in the sense of the Caratheodory theorem[1] and all cases when additional constraints are imposed are mentioned separately. The time dependence of all the quantities introduced is explicitly indicated in their definition and can later be omitted in order to reduce the size of the formulae, if this does not lead to misunderstandings.

We take a system of general form, described by the differential equation

$$\dot{x} = \phi(x, w, t), \quad t \geq t_0 \tag{1.1}$$

Here, x is the n-dimensional phase vector of the state of system (1.1), ϕ is the corresponding vector function, $w(t)$ is an m-dimensional noise vector (or the control vector of an opponent), t is the time (the independent variable) and t_0 is the initial instant. Suppose the vector $w(t)$ at any instant belongs to a certain known set in an m-dimensional space. We assume that the union of all possible initial states $x(t_0)$ is specified by the n-dimensional set X_0, that is, $x(t_0) \in X_0$.

The combination of the ends $x(t)$ of all the trajectories $x(\cdot)$ beginning at the instant t_0 at points of the initial set X_0 and satisfying the constraints imposed on the vector $w(t)$ is called the reachability set $\Upsilon(t) = \Upsilon(t, t_0, X_0)$ of system (1.1).

One of the main disadvantages of the guaranteed technique lies in the fact that, in the general case, operations on uncertain quantities change into operations on sets of an arbitrarily complex form. Even if the starting sets at the initial instant have a geometric form, requiring a small number of parameters for processing and storage and system (1.1) is linear with respect to x and w, manifolds of a complex and, most importantly, difficultly predictable form are obtained as a result of affine transformations, summation and intersection of the sets.

Estimates show that, even in the linear case, sufficiently complete pointwise descriptions of reachability sets in the high dimensional spaces of any of a wide class of real systems are practically impossible either now or in the foreseeable future without the creation of

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fundamentally new modelling devices. It is therefore natural to attempt to introduce sets of some single particular form that approximate real reachability sets and comparatively modest computational resources must be sufficient to attain an acceptable accuracy of the approximation. All the sets occurring in a problem change into sets of the specified form. Here, account has to be taken of the fact that changing from real systems to models frequently generates a considerable error compared with which the approximation error, when sets of a chosen form are used, can turn out to be unimportant. Two types of methods are commonly used:

1) those using parallelepipeds (see, for example, the review in Ref. 2), the faces of which are parallel to the coordinate planes in many cases. A change then occurs to coordinate-wise estimates within the framework of interval analysis, that is, a long-established and developed theory;[3–5]
2) those using ellipsoids.[6,7] This approach has been developed by many investigators and this paper is based on Chernousko's theory.[8]

From the point of view of its application in engineering, it is desirable that a guaranteed approximation technique should satisfy two conditions.

1°. The geometric figures, with which the reachability sets are finally approximated within the framework of some method, must be intelligible to the specialists who will make use of them.

A real application only needs a specific result of the calculations and this result can be obtained with satisfactory accuracy by many methods. A unique system of equations that are inherent in a technical device does not exist. Its constructor can be of great help to mathematicians in drawing up an adequate model of this device if he clearly imagines at this stage those objects with which theoreticians operate. In other words, it is desirable that, from the very start, the system of equations should be orientated toward a definite method of approximation.

For example, a useful software package[9] is specially adapted for the approximating of reachability sets by a large number of ellipsoids. However, in the absolute majority of these cases when an investigation has been carried out without the direct participation of the creators,[9] an approximation with either one ellipsoid[10–15] or two ellipsoids[16] has been used. The paper[17] is particularly interesting. In it, the required set was initially constructed using a software package[9] as the union of a large number of ellipsoids each of which lies within this set and touches the boundary of this set at a unique point. Considerable efforts were then made to reduce the number of the above-mentioned ellipsoids, replacing them with others that touched this boundary at two points. At the same time, it is seen (Ref. 3, Fig. 3) that the required set is indistinguishable from an ellipse.

2°. The method must allow of a comparison with a probabilistic estimate.

In spite of the fact that the stochastic and guaranteed approaches are quite different mathematically, both methods often can be used to describe one and the same technical device. This is explained by the fact that real perturbations practically never comprehensively satisfy the requirements of either the first or second methods. Hence, at the stage of changing over from an engineering problem to a mathematical problem, additional assumptions are introduced that can only very rarely be completely justified from a formal point of view.

For example, to accept a solution involving the use of probability theory, it is required, generally speaking, to verify the applicability of its axiom (see Ref. 18, for example) in a specific case which is exceedingly time-consuming even when this can be done with satisfactory accuracy. As a rule, we are limited to checking the stability of the frequencies, that is, to the requirement that the experiment should be reproducible an arbitrary number of times under practically identical conditions.[19] In turn, the guaranteed approach assumes, generally speaking, the realization of all physically possible versions of the development of events which is often equivalent to the presence of an opponent acting in an optimal way according to complex game theory (see Ref. 20, for example). In all cases, the degree of justifiability of the assumptions is measured as is customary by the magnitude of the errors that they introduce, taking the solution in favour of a specific method. Consequently, a choice has to be made between the guaranteed and probabilistic approaches. To do this, the guaranteed method must be "close"[21,22] to the probabilistic method and, as a rule, this last method rests on Gaussian distributions of the uncertain quantities. It is clear that the method of ellipsoids satisfies the requirement described best of all.

As an example, we will consider problems that are important for practical needs and require the processing of inexact measurements of the phase vector of a dynamic system with uncertainties. Mathematical models based on the stochastic approach are commonly used to refine the data from observations to be refined using knowledge of the properties of the system. The non-linear filters proposed by Kushner[23] and Stratonovich[24] possess the greatest possibilities. However, the volume of calculations required to implement them is extremely large (see Ref. 25, for example). This fact is one of the principal reasons why filters based on the Kalman filter[26] are mostly used.

2. Guaranteed analogue of a Kalman filter.

We will now present some basic results obtained earlier,[27] since these are required in the subsequent discussion, and consider the unsteady linear system

$$\begin{aligned} \dot{x} &= A(t)x + B(t)w(t), \quad x \in \mathrm{R}^n, \quad w(t) \in \mathrm{R}^m, \quad t \in [0, T] \\ y(t) &= H(t)x + \upsilon(t), \quad y(t) \in \mathrm{R}^r \end{aligned} \tag{2.1}$$

The matrices $A(t)$, $B(t)$ and $H(t)$ are known. The vector $w(t)$ describes a perturbation acting on the system and the vector $\upsilon(t)$ is the error in the observations. The exact values of the phase vector $x(t)$ and the corresponding observation vector are thereby unknown and the vector $y(t)$ belongs to a set known for each current instant. We also assume that system (2.1) when $w(t) \equiv 0$ and $\upsilon(t) \equiv 0$ is completely observable in the time interval considered.[28]

Suppose

$$E(\chi, \Theta) \tag{2.2}$$