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An exact solution to compute the band gap in phononic crystals

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ABSTRACT

In this paper, an alpha finite element method (αFEM) is formulated to study the propagation of elastic waves in periodic crystals. Using the simple triangular elements, the αFEM with optimal α value can always provide the exact solutions in the prediction of the band gap of phononic crystals. In addition, the upper and lower solutions are always observed in the computation of the band gap of phononic crystals with the adjustment of α value. The performance of αFEM is compared with standard finite element method (FEM) with different types of mass matrix formulation. The numerical examples have strongly demonstrated that the αFEM has given more accuracy prediction in the analysis of phononic crystals compared with standard FEM.

1. Introduction

Recently, a great deal of research has been devoted to the analysis of elastic wave in the phononic crystals with properties that are not normally found in nature [1]. With periodic variation in the density, elastic properties and structure, phononic crystals can exhibit pass or forbidden bands in their acoustic transmission spectrum, which leads to many practical applications of phononic crystals such as acoustic waveguides, sound and vibration isolators, and mechanical filters [2–5]. The band gaps in the phononic crystals is originated from Bragg reflections. In this case, the shielding for low frequency sound and vibration environment needs very large structures [1], which limits the application of phononic crystals.

Different from the Bragg type, the localized resonance of phononic crystals has demonstrated that the size of the periodic constant could be much smaller than the wavelength of the wave at the low frequency band gap [6,7]. The localized resonance of phononic crystals is firstly discovered by Liu et al. [8], who fabricated sonic crystals consisting of a soft material, a matrix, and a dense material [9] based on the idea of localized resonant structures. Lured by the excellent features of localized resonance of phononic crystals, various phononic crystals and their unique properties that are not observed in natural materials have been extensively explored and studied recently [4,10–13]. Their different microstructures with two or more constituents enable to achieve desirable properties in a way of multi-functionality and lightweight [14–17].

The further development in seeking new designs and applications of periodic phononic crystals needs an efficient, accurate and stable numerical methods. Some theoretical methods based on the plane-wave expansion (PWE) are employed to study the elastic wave of phononic crystals [18]. However, PWE may have some difficulties in dealing with phononic crystals with a large contrast in their elastic properties [19]. In addition, PWE has some inherent drawback in the case of lattice having a linear or point defect or a finite period lattice considered [20]. For these reasons, Finite Difference Time Domain (FDTD) is developed to overcome the disadvantages of PWE. Although FDTD offers tremendous insights on the phonon dispersion relations as well as transmission spectra [21], the treatment of boundary with complicated geometry is a bottleneck for the application of FDTD. Following this, Wang et al. [19] developed a lumped mass (LM) method based on FEM formulation to compute the band gap of phononic crystals. Compared with FDTD, LM shows some advantages in terms of convergence rate and computational efficiency.

It is well-known that the standard finite element method (FEM) suffer from the "overly-stiff" problem due to the use of compatible strain [22], which generally causes a significant loss of accuracy in the numerical results. This is particularly true for wave propagation problems, in which the overly-stiff stiffness causes the computed waves to propagate with artificially higher speeds than the actual ones in the media. This kind of dispersion error could be

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worse in the high frequency model [23,24], which leads to the poor prediction of elastic wave in the phononic crystals. Therefore, one possible improvement of solution is to soften the stiffness of the discretized model [25–28].

In this work, we present an efficient alpha finite element method (αFEM) [29,30] to further improve the predication of band gap of phononic crystals. The formulation of αFEM combines the overly-stiff FEM model and overly-soft node-based smoothed finite element method (NS-FEM) [26,31,32] with a parameter α. Compared with overly-stiff FEM, αFEM is a softened model that has a close to exact stiffness [33]. Thus, the accuracy from αFEM is much better than the standard FEM using the same number of triangular elements. Furthermore, there are several important properties in αFEM model. Firstly, the upper and lower bound solutions of the band gap generated by the overly-soft of the NS-FEM and overly-stiff of FEM [29] can be obtained with adjustment of α value, which is extremely important to estimate the exact solution using coarse mesh for designing phononic crystals. Secondly, with optimal α value, the exact solution of the band gap can be obtained. This is fantastic in the process of design of phononic crystals. Thirdly, the αFEM predicts more accurate solutions in comparison with standard FEM. In addition, the implementation of αFEM is very straightforward without changing FEM code too much.

The main goal of this paper is to present an efficient, stable and accurate algorithm to compute the stop band of phononic crystals. The detailed numerical results are provided to demonstrate the advantages of our proposed numerical scheme. Hopefully this advanced formulation of αFEM could be extended to solve more complicated problems in phononic crystals. The paper is outlined as follows. The brief introduction of elastic wave in the phononic crystals is described in Section 2. Section 3 discusses the formulation of αFEM. The numerical results are elaborated in Section 4. The conclusions are made in Section 5.

2. Band gap in phononic crystals

The governing equation of elastic wave in the isotropic media is described as follows:

$$\frac{\partial \sigma_{i1}}{\partial x_1}+\frac{\partial \sigma_{i2}}{\partial x_2}+\frac{\partial \sigma_{i3}}{\partial x_3}+F_i=\rho\frac{\partial^2 u_i}{\partial t^2}\quad i=1,2,3 \tag{1}$$

where σ denotes the stress, u stands for the displacement vector, ρ represents the density and F is the force.

The constitutive relationship for isotropic linear elasticity is expressed as follows:

$$\boldsymbol{\sigma}=\mathbf{D}\boldsymbol{\varepsilon} \tag{2}$$

where $\mathbf{D}$ is a fourth order tensor made up of Poisson's ratio and Young's modulus. The strain is given by:

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{3}$$

The matrix of material constants $\mathbf{D}$ can be expressed with shear and bulk modulus.

$$\mathbf{D}=\begin{bmatrix}\lambda+2\mu & \lambda & 0\\ \lambda & \lambda+2\mu & 0\\ 0 & 0 & \mu\end{bmatrix}\quad \text{2D plane strain} \tag{4}$$

where

$$\mu=\frac{E}{(1+2\nu)}\quad \lambda=\frac{\nu E}{(1+\nu)(1-2\nu)} \tag{5}$$

where μ and λ represent the shear and bulk modulus respectively, ν is the Poisson's ratio, and E is the Young's modulus.

A time harmonic form that is represented by a sine function as in equation:

$$u=u_0\sin\omega t \tag{6}$$

where ω is the angular frequency of mechanical motion.

$$\mu\nabla^2\mathbf{u}+(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})=-\rho\omega^2\mathbf{u} \tag{7}$$

Using standard Galerkin procedure with triangular element, Eq. (1) can be written in the matrix form:

$$[\mathbf{K}-\omega^2\mathbf{M}]\mathbf{u}=\mathbf{F} \tag{8}$$

where

$$\mathbf{K}=\int_\Omega \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\,\mathrm{d}\Omega\quad \text{stiffness matrix} \tag{9}$$

$$\mathbf{M}=\int_{\Omega^e}\rho\mathbf{N}^{\mathrm{T}}\mathbf{N}\,\mathrm{d}\Omega\quad \text{mass matrix} \tag{10}$$

In the standard FEM, the formulation of stiffness is obtained using compatible strain, leading to overly-stiff property of discretized model. In this work, the smoothed strain is introduced in the stiffness, which is elaborated in Section 3.

After assembly the stiffness and mass matrix, the implementation of boundary condition is an important step to compute the band gap of phononic crystals. It is noted that this step is exactly the same as standard FEM in the formulation of αFEM. According to Bloch's theorem, the relationships between the displacements, $\mathbf{u}$ and force $\mathbf{F}$ are obtained [34]:

$$\begin{aligned}
&\mathbf{u}_r=e^{ik_xa}\mathbf{u}_l,\quad \mathbf{u}_t=e^{ik_ya}\mathbf{u}_b\\
&\mathbf{u}_{rb}=e^{ik_xa}\mathbf{u}_{lb},\quad \mathbf{u}_{rt}=e^{i(k_x+k_y)a}\mathbf{u}_{lb},\quad \mathbf{u}_{lt}=e^{ik_ya}\mathbf{u}_{lb}\\
&\mathbf{F}_r=-e^{ik_xa}\mathbf{F}_l,\quad \mathbf{F}_t=-e^{ik_ya}\mathbf{F}_b\\
&\mathbf{F}_{rt}+e^{ik_xa}\mathbf{F}_{lt}+e^{ik_ya}\mathbf{F}_{rb}+e^{i(k_x+k_y)a}\mathbf{F}_{lb}=0
\end{aligned} \tag{11}$$

As outlined in Fig. 1(a), the left, right, bottom, top, and internal nodes of displacements are represented by $\mathbf{u}_l$, $\mathbf{u}_r$, $\mathbf{u}_b$, $\mathbf{u}_t$ and $\mathbf{u}_i$ respectively. The double subscripts denote the corner nodes of displacements.

Consequently, the following transformation can be defined as follows:

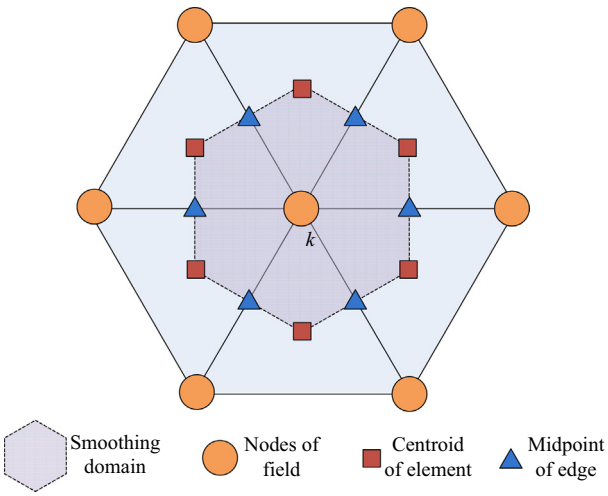

Fig. 1. Node-based smoothing domain.