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Canonical frame-indifferent transport operators with the four-dimensional formalism of differential geometry ☆

E. Rouhaud [a,*], B. Panicaud [a], R. Kerner [b]

[a] Université de Technologie de Troyes (UTT), Institut Charles Delaunay (ICD), CNRS UMR 6279, Systèmes Mécaniques et d'Ingénierie Simultanée (LASMIS), 12 rue Marie Curie, 10010 Troyes, France

[b] LPTMC, Tour 23-13, Boite Courrier 121, UPMC, 4 Place Jussieu, 75005 Paris, France

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ABSTRACT

To say that a constitutive model has to verify "the principle of material objectivity" to ensure its frame-indifference has become a common wisdom. Objective transports are thus defined to serve as tensor rates. These operators are in particular applied to the Cauchy stress tensor. They are used as time derivatives to describe non-linear or dissipative phenomena observed during the finite transformations of a material continuum. Because an infinite number of such transports may be constructed and shown to be objective, the selection of the appropriate transport and its validity still constitutes an open and debatable question.

Differential geometry, within its four-dimensional formalism, has proven its ability to describe physical fields and their variations in space and time while ensuring the covariance of any physical law. This description is here applied to the motion of a material continuum within the classical hypotheses of Newtonian physics. In this context, we show that the rate of a tensor as seen by a point of space–time is uniquely defined by the covariant rate; this quantity is not invariant with respect to superposed rigid body motions. The rate of a tensor as seen by a moving particle of matter is uniquely defined by the Lie derivative of the tensor. This operator is invariant with respect to superposed rigid body motions. Both, the covariant rate and the Lie derivative are independent of the observer and could thus be used in a constitutive model within a four-dimensional formalism. We show next that the projection of the Lie derivative of the Cauchy stress tensor within an inertial 3D Cartesian frame corresponds to Truesdell's transport and that the other 3D objective stress transports, if they have the dimension of a rate, do not correspond to a time derivative of this tensor. The Truesdell transport is thus the only objective transport that represents a frame-indifferent time derivative of the Cauchy stress tensor.

1. Introduction

The "principle of material objectivity" or "principle of material frame-indifference" plays an important role in continuum mechanics by constraining the formulation of material constitutive model [1]. A major issue corresponds to the definition of frame-indifferent transports to represent, objectively, the variations of a tensor with respect to time. These operators are indifferently referred to as objective rates, invariant time fluxes or objective transports. They are applied in particular to the Cauchy stress tensor. This issue has been first extensively discussed by Jaumann [2] who introduced an objective stress transport to serve as stress rate in constitutive models. Such objective rates are necessary to model complex solid or fluid materials [1,3–6]. They are also used within numerical formulations to solve non-linear problems.

The difficulty resides in the fact that there are *infinitely many possible objective time fluxes that may be used* as stated by Truesdell and Noll [1]. It is commonly admitted that the choice of the objective transport should be adapted to the kinematics of the modeled material but there is no specific mathematical or physical rule to guide this choice. Although Truesdell and Noll [1] postulate that *the properties of a material are independent of the choice of flux, which, like the choice of a measure of strain, is absolutely immaterial*, it is admitted that the transport operator could depend on the material to be modeled [5,7,8]. Objective stress transports have been interchangeably used for comparison, in particular in numerical computations [5,9–13]. The fact that several objective transports exist, leading to different results, to define a rate that should be kinematically meaningful, immaterial and frame-indifferent has lead several authors to pinpoint the difficulty with different interesting arguments [5,8,10–12,14–24].

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* Corresponding author. Tel.: +33 (0)32571806; fax: +33 (0)325715675.
E-mail address: emmanuelle.rouhaud@utt.fr (E. Rouhaud).

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Differential geometry proposes an algebra for tensors and their variations expressed in arbitrary coordinate systems. It is recognized as a formalism of choice to describe the straining motion of a material continuum within a 3D context; see, for example, [3,25,26,8]. Within its four-dimensional (4D) context, differential geometry has found a major and essential application in physics with the theory of General Relativity. This theory has shown its ability to deal properly with space–time transformations. Newtonian mechanics is embedded in General Relativity, as a limiting theory concerning phenomena for which the absolute speed of any points in negligible compared to the speed of light [27,28]. It has been shown that the formulation of Newtonian mechanics with a four-dimensional formalism offers an appropriate context to properly define objectivity and time derivatives [29–31]. We here propose to examine whether the transports used in continuum mechanics to construct constitutive models correspond indeed to actual time derivatives. We further verify whether these operators are frame-indifferent, in other word whether they satisfy the principle of covariance.

2. Problematics: kinematically meaningful transports

As proposed in [11], it is possible to list the characteristics that a transport operator should exhibit to be included in a constitutive model and/or within a numerical algorithm. One of the requirement is that this transport *should correspond to a time derivative*. Indeed, when constructing a material constitutive model in an incremental form, an instantaneous variation of the tensor over a time increment is needed; within a numerical algorithm, the discretization of this first order rate is needed for integration purposes. Fiala in [15] has initiated this approach, insisting on the necessity to find an expression for a *rate of change*.

Consider then a physical entity α. Further assume that this function depends on time t and on a set of other variables ξ themselves possibly being functions of time; both α and ξ are differentiable. The derivative of α with respect to time is given by:

$$\lim_{t'\to t}\frac{\alpha(\xi',t')-\alpha(\xi,t)}{t'-t} \tag{1}$$

with

$$\lim_{t'\to t}\xi' = \xi$$

The physical meaning of this derivative is given by the nature of α and the specific dependence of α and ξ on time. The fact that time is indeed considered as the physical variable for the differentiation makes of the above quantity a time derivative.

When the transport of a tensor is defined to serve as time derivative, it should be first written under the form of an increment following Eq. (1). Only then, should the corresponding transport operator be derived. This is what gives a kinematical meaning to this operator. Within a 3D context, it is necessary to choose a frame of reference to establish the classical time derivative operators such that these operators *depends* on the chosen frame. Objective transport operators have thus been proposed to solve this difficulty. Confusion arises because it is possible to define an infinite number of such operators. Further, it is not at all clear that these transports correspond to a time derivative although they are used as such. It is then difficult to choose one or the other on the basis of physical or kinematical considerations alone. The 4D formalism offers an opportunity to clarify these definitions within a fully frame-indifferent (covariant) context.

3. Four-dimensional description of space–time

Differential geometry (also known as Ricci-calculus) proposes a general mathematical context for the description of tensors and the associated algebra. The present Section introduces the definitions that are necessary for the rest of this work; a detailed presentation could be found for example in [32,33]. Classical notions of 4D physics are also reviewed to introduce specific vocabulary and notations. Details on these subjects are proposed for example in [34,35] where the general concepts are introduced, while the theory of General Relativity applied to physical fields is presented by Landau and Lifshits [36] and Weinberg [28].

3.1. Coordinates and their transformations

As opposed to classical mechanics, space–time is described with a four-dimensional continuous and differentiable manifold. The coordinates of a point within this manifold are parametrized by a set of four real numbers x^μ. This point is called an event and corresponds to a given position and instant of time. The coordinates are such that:

$$x^\mu = (x^1, x^2, x^3, x^4) = (x^i, x^4) \tag{2}$$

In this work, the index notation is used. The convention is such that Greek indices $\mu, \nu, \ldots$ run from 1 to 4 and label a four-dimensional entity. Latin indices i, j run from 1 to 3 and label the spatial part of this entity.

Other sets of coordinates could be indifferently chosen to parameterize the points of the manifold. Consider then, two possible sets of coordinates noted x^μ and $\tilde{x}^\mu$. The coordinate transformation from x^μ to $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$ implies that:

$$d\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dx^\nu \tag{3}$$

The matrix $\frac{\partial \tilde{x}^\mu}{\partial x^\nu}$ is the Jacobian matrix of the coordinate transformation and $\left|\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\right|$ is the determinant of this Jacobian matrix. Note that Einstein's summation convention has been used in Eq. (3) and will be used in the rest of this work.

3.2. Tensor densities

Tensor density fields of weight W can then be defined over the points of the manifold. For the sake of generality, they are noted $\boldsymbol{\alpha}$ to represent any given tensor density. Tensors densities are indifferent to an arbitrary change of coordinate systems or, equivalently independent to coordinate transformations. Thus, the components of a second rank tensor density $\boldsymbol{\alpha}$ always transform through a change of coordinates from x^μ to $\tilde{x}^\mu$ as [32,37]:

$$\tilde{\alpha}^\mu = \left|\frac{\partial x^\alpha}{\partial \tilde{x}^\beta}\right|^W \frac{\partial \tilde{x}^\mu}{\partial x^\nu}\alpha^\nu \tag{4}$$

$$\tilde{\alpha}^{\mu\nu} = \left|\frac{\partial x^\alpha}{\partial \tilde{x}^\beta}\right|^W \frac{\partial \tilde{x}^\mu}{\partial x^\lambda}\frac{\partial \tilde{x}^\nu}{\partial x^\kappa}\alpha^{\lambda\kappa} \tag{5}$$

$$\tilde{\alpha}_{\mu\nu} = \left|\frac{\partial x^\alpha}{\partial \tilde{x}^\beta}\right|^W \frac{\partial x^\lambda}{\partial \tilde{x}^\mu}\frac{\partial x^\kappa}{\partial \tilde{x}^\nu}\alpha_{\lambda\kappa} \tag{6}$$

It is possible to write similar equations for the components of tensor densities of any rank and, classically, upper indices denote the contravariant components of the tensor while lower indices denote its covariant components.