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A non-linear rod model for folded elastic strips

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ABSTRACT

We consider the equilibrium shapes of a thin, annular strip cut out in an elastic sheet. When a central fold is formed by creasing beyond the elastic limit, the strip has been observed to buckle out-of-plane. Starting from the theory of elastic plates, we derive a Kirchhoff rod model for the folded strip. A non-linear effective constitutive law incorporating the underlying geometrical constraints is derived, in which the angle the ridge appears as an internal degree of freedom. By contrast with traditional thin-walled beam models, this constitutive law captures large, non-rigid deformations of the cross-sections, including finite variations of the dihedral angle at the ridge. Using this effective rod theory, we identify a buckling instability that produces the out-of-plane configurations of the folded strip, and show that the strip behaves as an elastic ring having one frozen mode of curvature. In addition, we point out two novel buckling patterns: one where the centerline remains planar and the ridge angle is modulated; another one where the bending deformation is localized. These patterns are observed experimentally, explained based on stability analyses, and reproduced in simulations of the post-buckled configurations.

1. Introduction

Although the idea that a sheet of paper can be folded along an arbitrary curve is unfamiliar to many, performing this activity has been a form of art for quite some time. Bauhaus, the extinct German school of art and design, was a pioneer in developing the concept of curved folding structures by the end of the 1920s (Wingler, 1969). This practice often yields severely buckled and mechanically stiff sculptures featuring interesting structural properties and reveals new ways to think about engineering and architecture (Engel, 1968; Jackson, 2011; Schenk and Guest, 2011). Traditional origami has had a strong influence in the solution of many practical problems, to cite a few, the deployment of large membranes in space (Miura, 1980) and biomedical applications (Kuribayashi et al., 2006). However, exploring this long established art form still has a lot of potential. Since the work by Huffman (1976), an elegant and groundbreaking description of the geometry of curved creases, more attention has been devoted to this subject (Duncan and Duncan, 1982; Fuchs and Tabachnikov, 1999; Pottmann and Wallner, 2001; Kilian et al., 2008). A mechanical approach of structures comprising curved creases has recently been proposed (Dias et al., 2012) motivated by the intriguing 3d shapes shown in Fig. 1. In the present paper, we build upon this recent work by further exploring the mechanical models governing folded structures.

Folded structures combine geometry and mechanics: they deform in an inextensible manner and their mechanics is constrained by the geometry of developable surfaces (Spivak, 1979; do Carmo, 1976). Here, we consider one of the simplest

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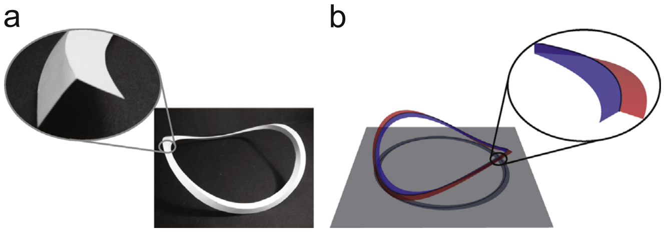

Fig. 1. Buckling of an annular elastic strip having a central fold: (a) model cut out in an initially flat piece of paper; (b) one of the goals of this paper is to represent this folded strip as a thin elastic rod, the ridge angle being considered as an internal degree of freedom.

folded structures: a narrow elastic plate comprising a central fold, as shown in Fig. 1. The role of geometry is apparent from the following observations, which anyone can reproduce with a paper model: the curvature of the crease line is minimum when the fold is flattened, a closed crease pattern results into a fold that buckles out of plane, while an open crease pattern results in a planar fold. These and other geometrical facts have been proved by Fuchs and Tabachnikov (1999).

The mechanics of thin rods has a long history (Dill, 1992; Love, 1944; Antman, 1995), and is used to tackle a number of problems from different fields today, such as the morphogenesis of slender objects (Wolgemuth et al., 2004; Moulton et al., 2013), the equilibrium shape of elongated biological filaments — such as DNA (Shi and Hearst, 1994) and bacterial flagellum (Powers, 2010) — and the mechanics of the human hair (Audoly and Pomeau, 2010; Goldstein et al., 2012). The classical theory of rods, known as Kirchhoff's rod theory, assumes that all dimensions of the cross-section are comparable: the consequence is that the cross-sections of the rod deform almost rigidly as long as long as the strain remains small. This assumption does not apply to a folded strip: its cross-sections are slender, as shown in the inset of Fig. 1(a), and, as a result, they can bend by a large amount. In addition, the dihedral angle at the ridge can also vary by a large amount.

Vlasov's theory for thin-walled beams overcomes the limitations of Kirchhoff's theory by relaxing some kinematic constraints and considering additional modes of deformations of the cross-section. This kinematic enrichment can be justified from 3d elasticity: assuming a thin-walled geometry, asymptotic convergence of the 3d problem to a rod model of Vlasov type has been established formally (Hamdouni and Millet, 2006, 2011). This justification from 3d elasticity requires that the deformations are mild, however: the cross-sections can only bend by a small amount away from their natural shape.

Mechanical models have been proposed to capture the large deformations of thin-walled beams. The special case of curved cross-sections must be addressed starting from the theory of shell: in this case, the bending of the centerline involves a trade-off between the shell's bending and stretching energies (Mansfield, 1973; Seffen et al., 2000; Guinot et al., 2012; Giomi and Mahadevan, 2012). By contrast, the strip that we consider is developable; it can be studied based on an inextensible plate model, in which the stretching energy plays no role. A model for a thin elastic strip has been developed Sadowsky (1930) in the case of a narrow ribbon, and later extended by Wunderlich to a finite width (Wunderlich, 1962). These strip models have found numerous applications recently, see Starostin and van der Heijden (2008) for instance. They have been developed independently of the theory of rods, as they make use of unknowns that are tied to the developability constraint.

Here, we develop a unified view of strips and rods. We show that elastic strips fit into the framework of thin rods: the equations for the equilibrium of a narrow, inextensible plate are shown to be governed by Kirchhoff's equations for an inextensible rod. To show this, we identify the relevant geometrical constraints and derive of an effective, non-linear constitutive law. A unified perspective of strips and rods brings in the following benefits: instead of re-deriving the equations of equilibrium for strips from scratch, which is cumbersome, we show that the classical Kirchhoff equations are applicable; we identify for the first time the stress variables relevant to the strip model, which is crucial for stability problems; the extension of the strip model to handle natural (geodesic) curvature, or the presence of a central fold becomes straightforward, as we demonstrate; stability analyses and numerical solutions of post-buckled equilibria can be carried out in close analogy with what is routinely done for classical rods.

This paper is organized as follows. In Section 2, we start by the smooth case, *i.e.* consider an elastic strip without a fold, and derive an equivalent rod model for it. In Section 3, we extend this model to a folded strip, which we call a bistrip; this is one of the main results of our paper. In Section 4, we derive circular solutions for the bistrip. Their stability is analyzed in Section 5, and we identify two families of buckling modes: one family of modes explains the typical non-planar shapes of the closed bistrip reported earlier, while the second mode of buckling is novel. The predictions of the linear stability analysis are confronted to experiments in Section 6, and to simulations of the post-buckled solutions in Section 7.

2. Smooth case: equivalent rod model for a curved elastic strip

We start by considering the case of a narrow strip having no central fold and show how it can be described using the language of thin elastic rods. The model we derive extends the model of Sadowsky (1930) to account for the geodesic curvature of the strip and bridges the gap between his formulation and the classical theory of elastic rods.