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Stability of strain-gradient plastic materials

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ABSTRACT

A formulation of Fleck and Willis (2009a,b) for strain-gradient plasticity has been adapted to provide possible descriptions for materials that initially strain-harden but eventually soften. In the absence of gradient terms, such material is unstable for any wavelength and subject to localization in the softening regime. Gradient terms do not mitigate the basic (infinite wavelength) material instability but they do inhibit the development of short-wavelength disturbances; they prevent localization but still may permit the development of narrow shear bands. In this work, the basic stability problem is studied via consideration of a small, generally time dependent, perturbation of an initially uniform state of deformation. The linearized problem for the perturbation is formulated for the general case of rate-dependent gradient plasticity but special attention is paid to the rate-independent limit. An interesting feature is that a qualitative difference is found between the effects of "energetic" and "dissipative" strain-gradient terms in this limit: energetic gradient terms permit the unbounded growth of any disturbance with wavelength larger than a critical value, whereas a disturbance of any finite wavelength in a medium with dissipative gradient terms can become unbounded only when the yield strength tends to zero.

1. Introduction

It is well-known that a classic rate-independent associative plastic material becomes unstable within the softening regime (Rice, 1976): the incremental equations governing the quasi-static problem are no longer elliptic while in dynamic conditions at least one wave-speed becomes imaginary. Under this condition any boundary value problem becomes ill-posed and the deformation localizes in a zero-thickness region (Bažant and Belytschko, 1985). For these reasons, any numerical analysis produces results that depend on the meshing procedure; in particular the width of the predicted localized zone is defined by the spacing of the discretization.

Needleman (1988) and Loret and Prévost (1991, 1993) showed that this problem can be resolved by admitting a viscosity parameter in the constitutive classic model (without the introduction of any internal characteristic length). The localization zone width then corresponds to the size of whatever imperfection was introduced in a static problem while it depends on the amount of viscosity in the dynamic problem (due to the consequent introduction of a non-microstructural internal length-scale).

Chambon et al. (1998) considered the localization within a one-dimensional problem for an elasto-plastic material characterized by non-local elasticity. They found that the introduction of a characteristic length does not lead

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automatically to the uniqueness of solution within the softening regime, but only a passage from infinite to a finite number of solutions.

Sluys et al. (1993) analysed the in-plane wave propagation problem, considering a yield function dependent on the second gradient of the plastic strain. They showed that the introduction of the characteristic length leads to a localized region with finite width and the dependence on the wavenumber of the wave propagation velocity (and in particular of the existence of its real part). Qualitatively similar results have also been obtained for gradient damage models (Peerlings et al., 1996; see also the review paper by de Borst, 2001).

Our purpose is to study the instability and the localization phenomena for non-local plastic material whose response falls within the framework set out by Gudmundson (2004) and developed further by Fleck and Willis (2009a,b), where the internal characteristic length can be described through either energetic and dissipative contributions. The novel feature that will emerge will be the qualitatively different influences of energetic and dissipative gradient terms. The usual bifurcation analysis that suffices in the case of classical plasticity (Anand et al., 1987; Bai, 1982) can be applied to the case of rate-independent gradient theory when the gradient terms are purely energetic but it becomes inapplicable in the presence of dissipative gradient terms. It becomes necessary to consider the development in time of a small perturbation. This can be pursued just as easily in the general case of rate-dependent material response. The resulting linear partial differential equations governing the perturbation have coefficients that depend on time, even in the rate-independent limit. The special case of the perturbation of a uniform monotonically increasing simple shear deformation is considered in detail.

2. Strain-gradient plasticity models

Consider a body occupying a domain V whose points are identified by the components of the position vector x_i (i=1,2,3). The total small strain tensor ε_{ij}, given as the symmetric part of the gradient of displacement vector u_i,

$$\varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2} \tag{2.1}$$

(where $,j$ represents $\partial/\partial x_j$), is additively decomposed into elastic and plastic parts

$$\varepsilon_{ij} = \varepsilon_{ij}^{EL} + \varepsilon_{ij}^{PL}. \tag{2.2}$$

Considering as fundamental kinematic quantities the elastic strain ε_{ij}^{EL}, the plastic strain ε_{ij}^{PL}, and its gradient $\varepsilon_{ij,k}^{PL}$ and defining as work-conjugate quantities, respectively, the symmetric Cauchy stress σ_{ij}, the generalized stress Q_{ij}, and the higher-order stress τ_{ijk}, the following principle of virtual work is postulated:

$$\int_V \{\sigma_{ij}\delta\varepsilon_{ij}^{EL} + Q_{ij}\delta\varepsilon_{ij}^{PL} + \tau_{ijk}\delta\varepsilon_{ij,k}^{PL}\}\, dV = \int_V f_i \delta u_i \, dV + \int_S \{\mathrm{T}_i \delta u_i + t_{ij}\delta\varepsilon_{ij}^{PL}\}\, dS, \tag{2.3}$$

where repeated indices imply summation, f_i represents the body-force per unit volume and T_i the surface traction (both work-conjugate to u_i) and t_{ij} represents the higher-order surface traction, work-conjugate to ε_{ij}^{PL}.

The principle of virtual work (2.3) implies the following equilibrium equations for points belonging to the volume V:

$$\begin{cases} \sigma_{ij,j} + f_i = 0 \\ Q_{ij} = (\mathrm{dev}\boldsymbol{\sigma})_{ij} + \tau_{ijk,k} \end{cases} \quad \text{in } V \tag{2.4}$$

(where dev represents the deviatoric part), and for points on the boundary S,

$$\begin{cases} \mathrm{T}_i = \sigma_{ij} n_j \\ t_{ij} = \tau_{ijk} n_k \end{cases} \quad \text{on } S. \tag{2.5}$$

In the case of dynamics, the first of Eqs. (2.4) is replaced by

$$\sigma_{ij,j} + f_i = \rho u_{i,tt}, \tag{2.6}$$

where ρ denotes the mass density; initial conditions have also to be specified. The main emphasis of this work is confined to the quasi-static case, in which inertia is disregarded; however some considerations about the dynamic case are made in Section 6.

Introducing the internal energy $U(\boldsymbol{\varepsilon}^{EL}; \boldsymbol{\varepsilon}^{PL}; \nabla\boldsymbol{\varepsilon}^{PL})$ we define "energetic" stresses as

$$\sigma_{ij}^{E} = \frac{\partial U}{\partial \varepsilon_{ij}^{EL}}, \quad Q_{ij}^{E} = \frac{\partial U}{\partial \varepsilon_{ij}^{PL}}, \quad \tau_{ijk}^{E} = \frac{\partial U}{\partial \varepsilon_{ij,k}^{PL}}, \tag{2.7}$$

leaving the "dissipative" contributions to be defined as follows:

$$\sigma_{ij}^{D} = \sigma_{ij} - \sigma_{ij}^{E}, \quad Q_{ij}^{D} = Q_{ij} - Q_{ij}^{E}, \quad \tau_{ijk}^{D} = \tau_{ijk} - \tau_{ijk}^{E}. \tag{2.8}$$