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Realizing the Willis equations with pre-stresses

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ABSTRACT

This paper proves that the linear elastic behavior of the material with inhomogeneous pre-stresses can be described by the Willis equations. In this case, the additional terms in the Willis equations, compared with the classical linear elastic equations for homogeneous media, are related to the gradient of pre-stresses. In this way, the material length scale is naturally incorporated in the framework of continuum mechanics. All these findings also coincide with the results of transformation elastodynamics, so that they can meet the requirement of the principle of material objectivity and the principle of general invariance.

1. Introduction

The classical linear elastodynamic theory was established for homogeneous media. In this theory, the elastic wave propagation is described by a constitutive equation

$$\boldsymbol{\sigma} = \boldsymbol{C}\colon \nabla \boldsymbol{u}, \tag{1a}$$

and an equation of motion

$$\nabla \bullet \boldsymbol{\sigma} + \boldsymbol{f} = \rho \ddot{\boldsymbol{u}}, \tag{1b}$$

where $\boldsymbol{\sigma}$ is the stress tensor; $\boldsymbol{f}$ is the body force vector; $\boldsymbol{C}$ is the elasticity tensor; ρ is the mass density; $\boldsymbol{u}$ is the displacement vector; and the superposed dot denotes the differentiation with respect to time t.

The elastodynamic equations for the linear elastic wave propagation in inhomogeneous media were established by Professor Willis over 30 years ago, based on the variational principle and the perturbation theory (Willis, 1981, 1997). The Willis equations are

$$\langle \boldsymbol{\sigma} \rangle = \boldsymbol{C}_{eff} * \langle \boldsymbol{e} \rangle + \boldsymbol{S}_{eff} * \langle \dot{\boldsymbol{u}} \rangle, \tag{2a}$$

$$\nabla \cdot \langle \boldsymbol{\sigma} \rangle + \boldsymbol{f} = \langle \dot{\boldsymbol{p}} \rangle, \tag{2b}$$

$$\langle \boldsymbol{p} \rangle = \boldsymbol{S}^{\dagger}_{eff} * \langle \boldsymbol{e} \rangle + \rho_{eff} * \langle \dot{\boldsymbol{u}} \rangle, \tag{2c}$$

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where ⟨⟩ denotes the ensemble average; $\boldsymbol{e}$ is the strain tensor, $\langle \boldsymbol{e} \rangle = (\nabla\langle \boldsymbol{u} \rangle + \langle \boldsymbol{u} \rangle\nabla)/2$; $\boldsymbol{p}$ is the momentum density; $\boldsymbol{C}_{eff}$, $\boldsymbol{S}_{eff}$ and ρ_{eff} are non-local operators depending on the angular frequency of oscillation; $\boldsymbol{S}^{\dagger}_{eff}$ is the adjoint of $\boldsymbol{S}_{eff}$; and * denotes the time convolution. In contrast to classical elastodynamic equations, the Willis equations can give more accurate predictions of wave behaviors in periodically inhomogeneous media (Willis, 2011; Norris et al., 2012; Srivastava and Nemat-Nasser, 2012). However, they have not been widely implemented for more general cases (Nassar et al., 2015), probably due to the unclear physical meaning and the abstract formulation of $\boldsymbol{S}_{eff}$.

In recent years, transformation optics (Leonhardt, 2006; Pendry et al., 2006) and transformation acoustics (Chen and Chan, 2010) have been widely used to obtain the inhomogeneously distributed effective material parameters of metamaterials (Shamonina and Solymar, 2007), which can steer the wave propagation along a desired trajectory. The basis of these transformation methods is the form-invariance of wave equations under general coordinate transformations. Unfortunately, Milton et al. (2006) pointed out that the classical elastodynamic equations are generally not form-invariant. And amazingly, if taking the deformation gradient as the gauge between the displacements before and after the transformation, one can obtain the Willis equations in frequency domain:

$$\boldsymbol{\Sigma} = \boldsymbol{C}:\nabla\boldsymbol{U} + \boldsymbol{S}\cdot\boldsymbol{U}, \tag{3a}$$

$$\nabla\cdot\boldsymbol{\Sigma} = \boldsymbol{S}^{\mathrm{T}}:\nabla\boldsymbol{U} - \omega^2\rho_{eff}\cdot\boldsymbol{U}, \tag{3b}$$

where $\boldsymbol{\Sigma}$ is the stress amplitude tensor; $\boldsymbol{U}$ is the displacement amplitude vector; $\boldsymbol{S}$ is a third-order tensor; the effective mass density tensor ρ_{eff} is an explicit function of angular frequency ω. These equations are form-invariant. In addition, all parameters inside are local if the microstructure of the material is sufficiently small compared with the interested wave length (Milton et al., 2006). Norris and Shuvalov (2011) further pointed out that the elasticity tensor $\boldsymbol{C}$ is generally non-symmetric when other gauges are adopted in the transformation.

However, according to *the principle of general invariance*, all laws of physics must be invariant under general coordinate transformations (Ohanian and Ruffini, 2013). I.e., form-invariance is an intrinsic property of correct wave equations, which should be expressed in tensorial forms. The requirement of *the principle of material objectivity* (Truesdell and Noll, 2004) in continuum mechanics could be regarded as a special case of *the principle of general invariance*, when the transformation is limited to an orthogonal mapping. According to these principles, Xiang (2014) pointed out that the proof of the form-invariance of wave equations does not need any assumption of the relation between field variables before and after the transformation. As a by-product, one can naturally obtain elastodynamic equations in time domain:

$$\boldsymbol{\sigma} = \boldsymbol{C}:\nabla\boldsymbol{u} + \boldsymbol{S}\cdot\boldsymbol{u}, \tag{4a}$$

$$\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{f}^{a} = \boldsymbol{S}^{\mathrm{T}}:\nabla\boldsymbol{u} + \boldsymbol{K}\cdot\boldsymbol{u} + \rho\cdot\ddot{\boldsymbol{u}}, \tag{4b}$$

where $\boldsymbol{\rho}$ is the mass density tensor; and $\boldsymbol{f}^{a}$ is the body force vector. The above equations are very similar to the Willis equations, if $\boldsymbol{C}$ is symmetric and $\boldsymbol{K}=\boldsymbol{0}$. In a special case when taking the deformation gradient as the gauge between the displacements before and after the transformation, $\boldsymbol{C}$ is the symmetric and the frequency version of Eq. (4) is exactly the same as Eq. (3) if ignoring the body force $\boldsymbol{f}^{a}$.

Based on the dimensional analysis of Eq. (4a), $\boldsymbol{S}$ can be intuitively regarded as the gradient of the pre-stress (Xiang, 2014). Since $\boldsymbol{K}$ has some relations with $\boldsymbol{S}$, it should have some relations with the gradient of the pre-stress. In addition, according to *the second law of thermodynamics*, an adiabatically separated system will become homogeneous eventually. Therefore, inhomogeneous materials must be the results of certain external forces, which may introduce the inside pre-stress. In this sense, the pre-stress is the concomitant of inhomogeneity. For example, all materials are strictly inhomogeneous due to the existence of interfaces or surfaces. Crystals near the interface or the surface are different from the crystals in other parts and consequently one can find interface or surface stresses (Müller and Saúl, 2004). And it is natural to read some reports about

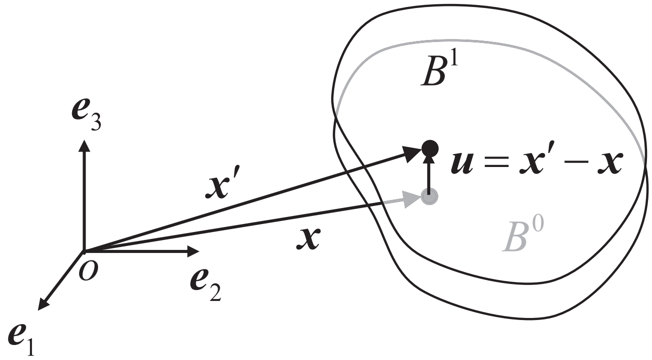

Fig. 1. The incremental model.