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A large-deformation thin plate theory with application to one-atom-thick layers

M.R. Delfani [a], H.M. Shodja [b,c,*]

[a] Department of Civil Engineering, K. N. Toosi University of Technology, Tehran, Iran
[b] Institute for Nanoscience and Nanotechnology, Sharif University of Technology, Tehran, Iran
[c] Department of Civil Engineering, Sharif University of Technology, Tehran, Iran

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ABSTRACT

Nowadays, two-dimensional materials due to their vast engineering and biomedical applications have been the focus of many researches. The present paper proposes a large-deformation theory for thin plates with application to one-atom-thick layers (OATLs). The deformation is formulated exactly in the mathematical framework of Lagrangian description. In particular, an exact finite strain analysis is given – in addition to the usual strain tensor associated to the middle surface, the second and third fundamental forms of the middle surface of the deformed thin plate are also maintained in the analysis. Exact closed-form solutions for a uniaxially curved thin plate due to pure bending in one case and due to a combination of vertical and horizontal loading in another are obtained. As a special case of the latter problem, the exact solution for the plane-strain bulge test of thin plates is derived. Subsequently, the approximation of Vlassak and Nix [Vlassak, J.J., Nix, W. D., 1992. J. Mater. Res., 7(12), 3242–3249] for the load–deflection equation is recovered. The given numerical results are devoted to graphene as the most well-known OATL.

1. Introduction

In recent years, one-atom-thick layers (OATLs) and especially graphene have opened a wealth of new venues in condensed-matter physics and materials science. Even though graphene had been known to scientists for decades, the unexpected finding of stable free-standing sheet of graphene as a truly two-dimensional crystal by Novoselov and Geim (2004, 2005) has brought it into the spotlight of researchers with diverse backgrounds within the past decade. Thanks to the exceptional properties of graphene, one can easily find a cornucopia of different applications for it (Geim and Novoselov, 2007). Moreover, Boron-nitride (BN) sheet can also be mentioned as another example of OATLs.

Among the attractive features of OATLs are their mechanical properties. In general, two different modes of the mechanical loading are of interest: in-plane and out-of-plane. To apply the in-plane mode of loading to a OATL, it can be subjected to uniaxial, biaxial or shear deformations. Wei et al. (2009), Cadelano et al. (2009), and Delfani et al. (2013) carried out first-principles calculations for monolayer graphene sheet under the in-plane states of deformation to evaluate its elastic moduli. Peng et al. (2012) carried out a similar study to determine the elastic moduli of BN sheet. Furthermore, OATLs can be subjected to the out-of-plane mode of deformation by applying concentrated load or uniform pressure. In practice, if it is intended to apply a point load on a sheet, an indenter is used to push it downward and, for uniformly distributed

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[*] Corresponding author at: Department of Civil Engineering, Sharif University of Technology, Tehran, Iran

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loading, a pressurized gas is introduced into a chamber beneath a suspended sheet that makes it bulged. In these tests, the obtained load–deflection curves are used to characterize the elastic behavior of the OATL and determine the desired parameters.

There exist many theoretical and experimental studies on the indentation and bulge tests of OATLs. For brevity, only a few of them are discussed herein. One of the most famous experiments on graphene has been conducted by Lee et al. (2008) who obtained its Young's modulus, third-order elastic stiffness, and intrinsic strength by nanoindentation using atomic force microscopy technique. In another experimental study, Lee et al. (2009) measured the elastic and frictional properties of graphene, respectively, by nanoindentation and friction force microscopy. Bunch et al. (2008) used the bulge test technique by applying a pressure difference across a graphene sheet to evaluate both its elastic properties and mass density. Atalaya et al. (2008) simulated both the elastostatic and elastodynamic behavior of a graphene sheet subjected to uniform pressure and harmonic driving force through utilizing an interatomic potential function. Duan and Wang (2009) carried out molecular mechanics simulations for the nonlinear bending and stretching of a circular graphene sheet under a central point load. Gil et al. (2010) investigated the formation of wrinkles and bulging in monolayer graphene sheets under nanoindentation and nanopressurization based on the minimization of a relaxed energy functional in conjunction with nonlinear finite hyperelasticity. Jun et al. (2011) utilized molecular dynamics to simulate the plane-strain bulge test of monolayer graphene sheet subjected to high gas pressure induced by hydrogen molecules. Fang et al. (2011) studied the mechanical behavior of graphene under indentation for various indentation depths, velocities, and temperatures using molecular dynamics analysis. Boddeti et al. (2013) carried out a study on the mechanics of pressurized graphene membranes using an experimental configuration that allows the determination of the elasticity of graphene and the adhesion energy between a substrate and a graphene membrane. Zhou et al. (2013), using molecular mechanics simulations, investigated the deformation mechanism of graphene sheet under free-standing indentation by considering the van der Waals interaction between indenter tip and graphene. Smith et al. (2013) proposed a pressure sensor based on the bulging mechanism of suspended graphene sheet and carried out tight-binding calculations of the graphene band structure as a function of strain. Jiang et al. (2014) simulated graphene sheets under surface pressure and point load as well as vibration and concluded that the deflection and fundamental frequency of graphene sheets are large enough to be utilized for high-sensitivity pressure sensors and resonators. Even though a major part of studies on OATLs has been devoted to graphene, BN sheets have been also the subject of some studies. For instance, the authors should mention an investigation by Song et al. (2010) who conducted experimental and molecular dynamics study on nanoindentation of BN films to measure their mechanical properties.

The experimental and simulation studies on the mechanical behavior of OATLs reveal that it is reasonable to model them as elastic lamina. The bending stiffness of such two-dimensional materials is negligibly small in comparison to their in-plane stiffness. In fact, their deformations are mainly accompanied by the membrane action. The theory of elasticity for membranes has been well-known for quite some time. The first studies date back to the early years of the twentieth century when Föppl and Hencky formulated the nonlinear elasticity of membranes (Föppl, 1907; Hencky, 1921; Föppl and Föppl, 1920). Adkins and Rivlin (1952) applied the theory of large deformations of incompressible isotropic materials to the problems involving the inflation of a circular membrane. The solution of some basic problems pertinent to the membrane theory is also available in the well-known works of Green and Zerna (1954) and Timoshenko and Woinowsky-Krieger (1959).Naghdi (1963) derived the general equations of linear theory of elastic shells and presented the corresponding first approximations under Kirchhoff–Love hypothesis. Green and Naghdi (1965) treated different problems associated with the linear theory of thin elastic shells. They presented membrane and inextensional theories and subsequently extended their developments to the dynamic problems. The works of Green et al. (1965) and Green and Naghdi (1968) are concerned with a general nonlinear dynamical theory of a surface with deformable directors which is referred to as a Cosserat surface. Green and Naghdi (1967) discussed some features of the linear theory of an elastic Cosserat plate and paid special attention to the bending of a transversely isotropic three-dimensional plate. Later, Green and Naghdi (1968) examined the relevance of the linearized theory of an elastic Cosserat surface to the classical problem of the linear theory of elastic shells, regarded as three dimensional bodies. Green and Naghdi (1974) mainly focused on the examination of the relation between a theory in which the shell is defined as a surface with a finite rotation vector and the theory of a Cosserat surface. They showed that the field equations of the former can be obtained as a special case of the general theory of a Cosserat surface. In the literature, various problems involving membrane theory have been proposed and treated by numerous investigators. For example, Dickey (1967) obtained the solution of the stress and strain field of a plane circular elastic surface subjected to a normal pressure. Callegari and Reiss (1968) proved the existence and uniqueness theorems of the boundary value problems for the axisymmetric deformation of a circular membrane subjected to a normal pressure by using the nonlinear Föppl membrane theory. Bhatia and Nachbar (1968) solved the nonlinear problem of an elastic sheet loaded transversely by a centered indenter with a hemispherical tip. Cohen and De Silva (1969) developed a nonlinear theory of elastic membranes and obtained the field equations and general constitutive relations of an elastic directed surface for both compressible and incompressible materials. Anderson and Arthurs (1970) from a mathematical point of view treated the nonlinear Föppl–Hencky differential equation and obtained approximate solutions by use of a simple class of trial functions. Yang and Hsu (1971) formulated the problem of axisymmetric deformations of an elastic membrane and applied the formulation to the problem of a circular flat membrane indented by a smooth sphere. Kao and Perrone (1971, 1972) employed a nonlinear relaxation method in conjunction with a finite difference approximation to solve the nonlinear partial differential equations governing large deformations of membranes. Jones (1974) obtained the governing differential equations for large