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Variational formulation and stability analysis of a three dimensional superelastic model for shape memory alloys

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ABSTRACT

This paper presents a variational framework for the three-dimensional macroscopic modelling of superelastic shape memory alloys in an isothermal setting. Phase transformation is accounted through a unique second order tensorial internal variable, acting as the transformation strain. Postulating the total strain energy density as the sum of a free energy and a dissipated energy, the model depends on two material scalar functions of the norm of the transformation strain and a material scalar constant. Appropriate calibration of these material functions allows to render a wide range of constitutive behaviours including stress-softening and stress-hardening. The quasi-static evolution problem of a domain is formulated in terms of two physical principles based on the total energy of the system: a stability criterion, which selects the local minima of the total energy, and an energy balance condition, which ensures the consistency of the evolution of the total energy with respect to the external loadings. The local phase transformation laws in terms of Kuhn–Tucker relations are deduced from the first-order stability condition and the energy balance condition.

The response of the model is illustrated with a numerical traction–torsion test performed on a thin-walled cylinder. Evolutions of homogeneous states are given for proportional and non-proportional loadings. Influence of the stress-hardening/softening properties on the evolution of the transformation domain is emphasized. Finally, in view of an identification process, the issue of stability of homogeneous states in a multi-dimensional setting is answered based on the study of second-order derivative of the total energy. Explicit necessary and sufficient conditions of stability are provided.

1. Introduction

Shape memory alloys (SMA) are materials which exhibit non-conventional thermo-mechanical properties, with two of the most remarkable being *pseudoelasticity* and *shape memory effect*. While the former refers to the capability of the material to regain the initial strain after a loading–unloading hysteretic cycle, the latter is related to the recovery of initial shape after occurrence of “apparently” permanent strains, simply by heating the material above a critical temperature. Both phenomena are associated to complex phase transitions at both micro- and macro-scale.

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Nomenclature

E_0	elastic Young's modulus
$\mu(y)$	elastic shear modulus as material function
K	elastic bulk modulus
$\bar{\sigma}_0$, $\Delta\sigma$	material constants
σ_{Fw}, σ_{Rv}	forward and reverse phase transformation stresses
$\bar{\sigma}(y)$	critical phase transformation stress as material function
p_1	material function
$\boldsymbol{u}$	displacement
ε, $\boldsymbol{\varepsilon}$	(total) strain
σ, $\boldsymbol{\sigma}$	stress
$\boldsymbol{\varepsilon}^D$, $\boldsymbol{\sigma}^D$	deviatoric strain and stress
p, $\mathbf{p}$	phase transformation strain
$\bar{\mathrm{p}}$	cumulated phase transformation strain
N	dimension of the space
Ω	reference configuration of the N-dimensional body
W	total strain energy density
$\mathcal{E}$	total energy
$\mathcal{R}$	Rayleigh ratio
$\mathcal{C}$, $\mathcal{C}_0$	admissible spaces of displacement and displacement rate fields
$\mathcal{Z}$, $\mathcal{Z}_0$	admissible spaces of phase transformation strain and strain rate fields
$\bar{\mathcal{Z}}$	admissible space of cumulated phase transformation strain field

Due to these properties, polycrystalline SMA have been extensively studied with a great scientific impact and today are widely used for several advanced engineering applications like, among the others: biomedical devices, aerospace structures, mechanical components, MEMS as well as civil engineering structural elements or devices (Mohd Jani et al., 2014 for a recent review of SMA about research, applications and opportunities). One of the key requirement of superelastic SMA is the size of its hysteresis loop. Indeed, a large hysteresis loop will lead to a substantial capability to dissipate energy. This is particularly important when SMA are used as dampers in civil engineering. Experimental investigations show that SMA with large hysteresis loop correspond to phase transformation occurring with a stress plateau and a *localized* phase transformation front (Shaw and Kyriakides, 1995). However, a large number of loading cycles leads, as a result, to a smaller hysteresis loop with weakened dissipative properties and to an increasingly stress-hardening behaviour associated to a *homogeneous* phase transformation (Moumni et al., 2009).

The origin of macroscopic localization and stress plateau in SMA is related to an intrinsic *stress-softening* behaviour. Such kind of behaviour can be found in other type of dissipative phenomenon, such as damage, with similar consequences, e.g. macroscopic instabilities and non-homogeneous evolutions (Pham et al., 2011). On the other hand, *stress-hardening* has a stabilizing effect and is usually associated to a unique diffuse response. Experimentally, such stress-softening behaviour leads to important issues in the identification of the material parameters of the model (Churchill et al., 2009; Iadicola and Shaw, 2002). Indeed, a classical tensile test on a SMA specimen provides direct information only if the response is homogeneous, so that the material behaviour can be read through the structural response (León Baldelli et al., 2015; Alessi and Bernardini, 2015). Stress-softening and macroscopic instabilities limit the use of such tensile test and non-conventional experimental setups are required to overcome such issues (Iadicola and Shaw, 2002). The main idea in the recent experiments of Hallai and Kyriakides (2013) is to force a homogeneous transformation, despite stress-softening, by bonding in parallel two strips, properly designed, of stress-hardening stainless steel with a softening SMA core strip. As a result, the global behaviour of the laminate exhibits a structural hardening response which is associated with a diffuse phase transformation. By subtracting the known and well-defined response of the stainless steel, one has access to the intrinsic behaviour of the SMA. The result is an up-and-down material response with a stress-hardening part only for a very short nominal strain range ($<1\%$) followed by a large stress-softening part ($\sim 5\%$). Due to residual plastic strains in the stainless steel strips, only the forward intrinsic phase transformation has been in this work extracted.

Another important feature of SMA is related to their macroscopic response under multi-axial loading. Specifically, although the literature is less dense than when it comes to uniaxial loading, several authors (Sittner et al., 1995, 1996; Lim and McDowell, 1999; Bouvet, 2001; Helm and Haupt, 2001; Sun and Li, 2002; Feng and Sun, 2006; Echchorfi, 2013) have studied the SMA behaviour in the multi-axial context with traction–torsion tests on thin-walled cylinders, where they highlighted a complex evolution of the specimen due to the reorientation of the martensite variants towards the direction of the mechanical loading. In particular, a dissymmetry in the material response has been observed with a stress-hardening homogeneous response in pure shear and a stress-softening response with localizations in simple tension. Unfortunately, few information is given on the transformation domain and its evolution in terms of shape and size except for the onset of the phase transformation. As in plasticity, the knowledge of such transformation domain is however fundamental for a rigorous construction of a phenomenological model. Another well established experimental evidence is that the mechanical response is almost independent with respect to hydrostatic pressure, hence with respect to the first invariant of the stress tensor. Indeed, as pointed out in the experiments by Gall et al. (1998), pure hydrostatic pressure was unable to trigger the martensitic transformation at pressures of up to 700 MPa.

A consistent macroscopic model of SMA should then be able to account for all the aforementioned main characteristics, namely stress-softening, reorientation and incompressibility of the phase-transformation. Different classes of macroscopic