An energy-deformation decomposition for morphoelasticity

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ABSTRACT

Mathematical models of biological growth commonly attempt to distinguish deformation due to growth from that due to mechanical stresses through a hypothesised multiplicative decomposition of the deformation gradient. This multiplicative decomposition is valid only under restrictive hypothesis, and can fail in many instances of scientific relevance. Shifting the focus away from the kinematics of growth to the mechanical energy of the growing object enables us to propose an "energy-deformation decomposition" which accurately captures the influence of growth on mechanical energy. We provide a proof and computational verification of this for tissues with crystalline structure. Our arguments also apply to tissues with a network structure. Due to the general nature of these results they apply to a wide range of models for growing systems.

1. Introduction

1.1. Background

Biological growth—of cells, tissues, organs and organisms—leads to morphological change as well as mechanical stresses such as tension in arteries (see, for example, Holzapfel and Ogden, 2010) and plant stems (see, for example, Goriely and Neukirch, 2006; Vandiver and Goriely, 2008). These play an important role in biological function. Modeling biological growth and the accompanying mechanical stresses is of increasing interest in the biological, continuum mechanics and mathematical communities as it becomes abundantly clear that mechanical stresses are not only passive responses to growth, but also feed back to influence morphological development as well as biochemical pathways; see, for example, Lintilhac and Vesecky (1984), Lynch and Lintilhac (1997), and Huang (2004).

Continuum models for growth typically draw from approaches first developed in the context of non-biological continua, for example, plasticity theory or mixture theory; see Goriely et al. (2008) and Ambrosi et al. (2011) for reviews. A very popular current approach is based on a multiplicative decomposition of the deformation gradient. The idea, introduced by Rodriguez et al. (1994), is as follows: suppose that a map φ describes the deformation of a body from a reference configuration to a current configuration. In the absence of growth, the elastic energy density of the current configuration is a function W of the deformation gradient $F:=D\varphi$. If, however, growth has also contributed to the deformation, the mechanical energy is a function of only that part of the deformation that results in elastic stresses. Analogous to a standard approach in plasticity theory (Lee, 1969), the approach of multiplicative decomposition posits that the deformation gradient F is a

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product (point-wise in space–time)

$$F = A\,G \tag{1a}$$

of tensors G, arising from growth, and A, arising from elastic deformation. Because there is not, in general, a deformation of the body with gradient G—that is, growth may not be compatible with any actual deformation—the elastic deformation associated with A restores compatibility (that is, the gradient nature) of F. The elastic energy density is a function only of A

$$W = W\left(FG^{-1}\right). \tag{1b}$$

1.2. Main results

From (1b), the multiplicative decomposition can be understood as saying that energetically growth is a *motion* (in its domain, by G) of the elastic energy density W. While this is sometimes true (see Lemma 4.4 below) the focus of this paper is on circumstances where a subtler approach is called for. As we shall see (Remark 4.7 below), these are situations where shear resistance (see Definition 1.1 below) is present and growth is non-dilatational (defined in Section 1.4 below).

We propose a way to apply the essential insights of multiplicative decomposition in this broader setting: that in addition to the deformation gradient being decomposed (multiplicatively), the energy density be decomposed (additively). This is done in Theorems 4.1 and 4.14, and Corollary 5.1 below. These focus on growing systems with a crystalline structure; we explain the reason for this in Section 1.3. However, our insights and arguments are more general and we comment on growing networks and growing continua in Section 6. We highlight three features of our approach.

First, we study continua via discrete systems: we begin with discrete systems because they provide a context in which the concept of growth can be clearly and unambiguously formulated, and where the interaction of growth with deformation can be rigorously derived. To understand growth on a continuum level, we compare the continuum limits of the initial and the grown system. (However, in some instances, discrete systems themselves are appropriate models for growth with the discrete elements corresponding, for instance, to biological cells, see, for example, Odell et al., 1981; Weliky and Oster, 1990; Muñoz et al., 2010; Jones and Chapman, 2012.)

Second, we shift attention away from the kinematics of growth, cf., (1a), to the energetics of growth, cf., (1b). We submit that not only is this a better conceptual approach to growth but also facilitates the use of more sophisticated mathematical tools such as discrete-to-continuum limits and, more generally, the tools of the calculus of variations. In addition it enables a closer integration of the mechanics of growth with the biochemistry of growth, and thus provides a better framework for the development of more holistic models of biological growth.

Third, we recognise that there are different kinds of growth processes, two of which we discuss in Section 2 below.

1.3. Organisation of the paper

We begin, in Section 2, by considering *growable springs*. These are one-dimensional elastic objects which are able to change their rest length by a non-elastic process, namely *growth*. There is more than one way in which this can occur; we present two ways, which we name *replication* (Section 2.1) and *recombination* (Section 2.2). Of these replication, which involves mass transfer, more naturally corresponds to an intuitive understanding of growth. However, recombination, which is a constant-mass process, may be thought of as a one-dimensional conceptualisation of the biological process sometimes called "remodelling", see, for example, Taber (1995), Taber (2001), Ambrosi et al. (2011), and Menzel and Kuhl (2012). Recombination is also simpler to analyse mathematically so we focus on it in this paper. However, as a reader who follows our arguments can easily see, our comments on the limitations of multiplicative decomposition are true for generic growth processes including replication. Similarly, while we develop the approach of "energy-deformation decomposition" in the context of recombination, the insight underlying it applies to generic growth processes (albeit the resulting formulation might be less elegant for replication).

In Section 3.1 we introduce (node–spring) *lattices*, which provide our model of a biological system. We do this, not because lattices are good models for biological systems but rather because they present the simplest context in which we can communicate our insights and develop our arguments. It would have been biologically more natural to use (node–spring) *networks* but we judged that the resulting need for more detailed mathematical analysis would have obscured rather than clarified the essential features. After this, in Section 3.2, we introduce a one-dimensional example which serves as a concrete context in which to introduce the questions that would occupy our attention.

Next we come to the heart of the paper: In Section 4 we present our main results (in the context of homogeneous lattices growing homogeneously) and arguments (which are much broader in scope). Section 5 extends these results to inhomogeneous situations by presenting computational evidence that the energy-deformation decomposition is stable under homogenisation, both of growth and of elasticity. While Sections 4 and 5 considered continuum limits of lattices, in Section 6 we briefly touch upon the application of our insights to networks and continua.