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Straightening wrinkles

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ABSTRACT

We consider the elastic deformation of a circular cylindrical sector composed of an incompressible isotropic soft solid when it is straightened into a rectangular block. In this process, the circumferential line elements on the original inner face of the sector are stretched while those on the original outer face are contracted. We investigate the geometrical and physical conditions under which the latter line elements can be contracted to the point where a localized incremental instability develops. We provide a robust algorithm to solve the corresponding two-point boundary value problem, which is stiff numerically. We illustrate the results with full incremental displacement fields in the case of Mooney–Rivlin materials and also perform an asymptotic analysis for thin sectors.

1. Introduction

The back of the elbow on an extended arm and the region of contact between the road and a tyre are two examples of a *straightening deformation*. In this paper we revisit and take further the analysis of this somewhat neglected exact solution of nonlinear elasticity. We extend it to include a study of *incremental instability*: in the case of the elbow, instability could model the appearance of wrinkles; in that of the tyre, it would correspond to the onset of the so-called Schallamach (1971) stripes; see Fig. 1 for photographs.

There are only six known families of large deformations which are universal to all nonlinear incompressible isotropic materials (see for instance the textbook Tadmor et al., 2012 for a recent exposé). They are

- *Family* 0: Homogeneous deformations.
- *Family* 1: Bending, stretching and shearing of a rectangular block.
- *Family* 2: Straightening, stretching and shearing of a sector of a hollow cylinder.
- *Family* 3: Inflation, bending, torsion, extension and shearing of an annular wedge.
- *Family* 4: Inflation or eversion of a sector of a spherical shell.
- *Family* 5: Inflation, bending, extension and azimuthal shearing of an annular wedge.

There are countless stability studies for *Families* 0, 1, 3, and 4; however, there exists no work on the stability of a straightened sector. In fact, the literature on the large straightening deformation itself is very sparse and we have been able

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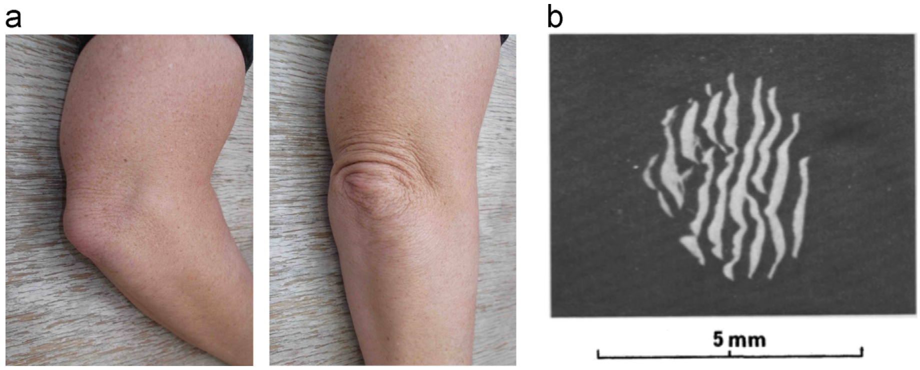

Fig. 1. Straightening instabilities. (a) As the arm extends, wrinkles may appear on the surface of the elbow, the number of which seems to depend mostly on the age of the subject. (b) Single frame of a film of the contact between perspex and a butyl sphere sliding over it at 0.043 cm/s, showing the appearance of instability stripes (taken from Schallamach, 1971.

to identify only five contributions on the subject (Ericksen, 1954; Truesdell and Noll, 2004; Aron et al., 1998; Aron, 2000, 2005).

Although in practice, bending (*Family* 1) and straightening (*Family* 2) are the opposite of each other, their respective theoretical modelling differs completely. For instance, we shall see in this paper that the large straightening deformation occurs in plane strain and in plane stress ($\lambda_3 = 1$ and $\sigma_1 = 0$ throughout the block), while large bending is accompanied by plane strain only (the normal stress is zero only point-wise on the bent faces). That no result at all can be deduced from one deformation to apply to the other is particularly true for the stability analysis, which proves to be extremely difficult to conduct. Both problems can be formulated in terms of a linear ordinary differential system with variable coefficients which is stiff numerically. For the bending problem, this numerical stiffness can be smoothed out by the Compound Matrix method, which has proved to be most successful in the past in stability analyses for *Family* 0 (e.g. Haughton, 2011), *Family* 1 (e.g. Haughton, 1999; Coman and Destrade, 2008; Roccabianca et al., 2011), *Family* 3 (e.g. Destrade et al., 2010) and *Family* 4 (e.g. Fu, 1998; Fu and Lin, 2002). Here we tried several numerical methods in turn for the straightening problem: Determinantal method, Compound Matrix method, Surface Impedance method, and only the latter one turned out to be robust enough to handle this very stiff problem (see also Destrade et al., 2009, 2014).

In his seminal paper on "Deformations Possible in Every Isotropic, Incompressible, Perfectly Elastic Body", Ericksen (1954) showed that a circular sector can always be "straightened" into a rectangular block, by mapping the reference cylindrical polar coordinates (R, Θ, Z) of the material points in the region

$$0 < R_1 \le R \le R_2, \quad -\Theta_0 \le \Theta \le \Theta_0, \quad 0 \le Z \le H, \tag{1}$$

to the rectangular Cartesian coordinates (x_1, x_2, x_3) of points in the region

$$a \le x_1 \le b, \quad -l \le x_2 \le l, \quad 0 \le x_3 \le H, \tag{2}$$

in the deformed configuration through the deformation

$$x_1 = \frac{1}{2}AR^2, \quad x_2 = \frac{\Theta}{A}, \quad x_3 = Z; \tag{3}$$

see Fig. 2 and also Truesdell and Noll (2004).

Here $0 < 2\Theta_0 < 2\pi$ is the angle spanning the solid sector, and $2\pi - 2\Theta_0$ is what we will call the "open angle" of the original, undeformed, sector (in contrast to the term "opening angle" of a deformed sector used, for example, in Destrade et al., 2010). Also, $R = R_1$ and $R = R_2$ are the inner and outer faces of the open sector, respectively, while $x_1 = a$ and $x_1 = b$ are their counterparts in the deformed rectangular block. The quantity A, which has the dimensions of the inverse of a length, is to be determined from the boundary conditions. In general, its value depends on the loads imposed on the end faces $x_2 = \pm l$ of the straightened block to sustain the deformation. Examples of such loads (in the absence of body forces) include a resultant normal force N, a resultant moment M, or a mixture of the two. In this paper we take the point of view that the final length $2l$ of the block is found from the stability analysis and we calculate the corresponding A, N and M required to attain it. In particular, it is clear from (3) that

$$A = \Theta_0 / l, \tag{4}$$

and that the faces normal to the x_1 axis are at

$$x_1 = a = \frac{1}{2}AR_1^2 = \frac{\Theta_0}{2l}R_1^2, \quad x_1 = b = \frac{1}{2}AR_2^2 = \frac{\Theta_0}{2l}R_2^2. \tag{5}$$

Section 2 covers the characteristics of the large straightening deformation. In particular, we show there that the principal Cauchy stress component σ_1 is zero throughout the block.