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Strain induced shape formation in fibred cylindrical tubes

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ABSTRACT

The aim of the paper is to solve and discuss the representation problem of a special class of integrable distortion fields in fibred cylindrical bodies and to analyze the corresponding induced shape changes. We find and discuss the compatibility conditions, *i.e.* the conditions to be satisfied to get a pair compatible distortion/shape change, when different fields of fibers are assigned on the cylindrical body, through the specification of the fields of fiber angles.

1. Introduction

Usually, in the language of solid mechanics, the changes suffered by the bodies as a consequence of inelastic effects are modeled through the notion of distortion, which measures locally the changes in the zero-stress state of the material. Elasticity theory has long since accounted for the effects of distortions, and a large body of literature is devoted to the analysis of the residual stresses induced by thermal strains, consequence of heating (cooling) metals or other typical construction materials, especially in the small strain regime (Boley and Weiner, 1960). More recently, starting from the pioneering works of Rodriguez et al. (1994) and Taber (1995), focus has been set on soft matter which is particularly prone about noticeable morphological changes, which may arise as a consequence of volumetric growth, remodeling, and phase transitions, and may be described within the theory of non-linear, inhomogeneous, anisotropic elasticity with distortions. In the last decade, many efforts have been done in the development of that theory, along three main research lines: theoretical foundation, physical applications and, in between, general investigation. Within the papers devoted to the analysis of the theoretical foundations of the growth theory as a special case of the finite elasticity theory whit distortions, we cite the ones by Epstein and Maugin (2000), DiCarlo and Quiligotti (2002) and Lubarda (2004). A lot of physical applications of the finite elasticity with distortions may be found in the recent scientific literature on soft matter concerning swelling or shrinkage of hydrogels (Wallmersperger et al., 2011), isotropic-nematic transitions (Sawa et al., 2010), voltage-induced deformations in ionic polymer–metal composites (Nemat-Nasser, 2008), growth of plants (Burgert et al., 2007; Vandiver and Goriely, 2008), tissue remodeling (Rodriguez et al., 1994), muscle activation (Nardinocchi and Teresi, 2007), remodeling of arteries (Taber and Humphrey, 2001). In each of these cases the distortions are ruled by different physics which, in a few cases, are described through the appropriate balance laws giving so origins to the so-called multiphysics models. General investigations on stability issues related to volumetric growth in soft materials

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have been recently carried out in Sharon et al. (2007), Marder et al. (2007), Goriely and Ben Amar (2005),Ben Amar and Goriely (2005), Dervaux and Ben Amar (2008) and Moulton and Goriely (2011). In the last group of papers, stability issue and volumetric growth are discussed together and the effect of the growth incompatibility on the stability of the elastic solution is studied for bodies with simple geometries.

In particular, in Goriely and Ben Amar (2005) and Ben Amar and Goriely (2005) both inhomogeneous and isotropic growth, and anisotropic and homogeneous growth are identified as the ones inducing incompatible distortions. Hence, the elastic problem is studied with reference to a body with spherical symmetry with the aim to discuss the dependence of the stability issues on the growth and on the correspondingly induced stress state. In Dervaux and Ben Amar (2008) the stability of a flat disk under the effect of an anisotropic growth is studied. It is concluded that the disk can buckle out of its plane toward a stress-free or quasi stress-free configuration with a prescribed Gaussian curvature as the higher dimension of the embedding space allows the body to solve (or quasi) the incompatibility (Ciarlet, 2005). In Moulton and Goriely (2011) the same studies are performed on a thick cylinder; in particular, anisotropic and (piecewise) inhomogeneous distortions are discussed.

The common element of these studies is the focus on stability issues and its relation with both the stress state induced by incompatible growth distortions and the amount of the growth distortion. In the present paper, we focus on the existence and possible representation of compatible distortions. Of course, as in the above discussed literature, to carry on explicit representation formulas, body with a specific symmetry is needed. Nevertheless, we admit that the distortion field whose compatibility has to be discussed, do not have the same specific symmetry of the body, as it is usually done through the choice of distortions sharing the symmetry group of the body (Goriely and Ben Amar, 2005; Ben Amar and Goriely, 2005; Dervaux and Ben Amar, 2008; Moulton and Goriely, 2011). Of course, the case of Moulton and Goriely (2011), where distortion with cylindrical symmetry are considered, may be recovered as a special case. Furthermore, we admit inhomogeneous and anisotropic distortion fields as inhomogeneity can compensate anisotropy in order to lead to compatible distortions; and to view inhomogeneity and anisotropy as two independent causes of incompatibility may strongly affect the result.

The (inverse) elastic problem we solve and discuss may be sketched as follows: given a stress free cylindrical body with no loads, to look for the distortion field which has a few general characteristics and leads to a new stress free configuration. The solution of this problem is not unique and the discussion of a characteristic system of equations, allows us to characterize a specific class of distortions.

The paper is organized as follows: firstly the theory of non-linear elasticity with distortions is briefly presented together with the characteristics of the cylindrical body and fibred distortion field; the representation problem is solved and general solution for fibred distortion fields is found; the solution is analyzed in a few special relevant cases and discussed.

2. Distortions and compatibility in non-linear elasticity

Within the theory of elasticity, *distortions* are introduced to model a *change in the zero-stress state*, or relaxed state, of a material body. With this in our mind, let $\mathcal{B}$ be the body manifold, identified with its reference configuration, $\mathcal{E}$ be the 3D Euclidean ambient space, with $\mathcal{V}$ the associated vector space, and $\mathbb{L}\mathrm{in}=\mathcal{V}\otimes\mathcal{V}$ be the space of double tensors on $\mathcal{V}$ (linear maps of $\mathcal{V}$ into itself). A mechanical state is described by two smooth fields, the placement χ and the distortion $\mathbf{F}_o$

$$\chi:\mathcal{B}\to\mathcal{E},\quad \mathbf{F}_o:\mathcal{B}\to\mathbb{L}\mathrm{in},$$

$$X\mapsto x=\chi(X),\quad X\mapsto\mathbf{F}_o(X) \tag{2.1}$$

associating to any material point $X\in\mathcal{B}$ its *position* in space $x=\chi(X)\in\mathcal{E}$, and its *relaxed stance* $\mathbf{F}_o(X)$. The subtle role of the notion of distortion is highlighted by quoting *verbatim* from the paper (DiCarlo and Quiligotti, 2002): "Notice that the actual placement χ and the relaxed stance $\mathbf{F}_o$ belong to two different layers of description. While the introduction of χ preceeds dynamics altogether, $\mathbf{F}_o$ cannot even be conceived without the standard notion of stress and some constitutive information on it." To grasp the notion of distortions, let us consider a body volume-element dV such that $dV_o=\mathbf{F}_o\,dV$ represents its relaxed state and $dv=\mathbf{F}\,dV$ its actual state; these two instances of a body volume-element are in turn related by a tensor field $\mathbf{F}_e$, see Fig. 1, such that

$$\mathbf{F}=\mathbf{F}_e\mathbf{F}_o. \tag{2.2}$$

Eq. (2.2) represents the multiplicative decomposition (Kröner, 1960; Lee, 1969) of the deformation gradient $\mathbf{F}$ into the *elastic deformation* $\mathbf{F}_e$ and the inelastic distortion $\mathbf{F}_o$. The actual configuration of the body is described by $\chi(\mathcal{B})$; in general, a relaxed configuration cannot be realized, not even locally, and thus, it cannot be described by a placement: precisely, the symmetric tensor $\mathbf{C}_o=\mathbf{F}_o^T\mathbf{F}_o$ may not be the *metric tensor* of any possible configuration.

It happens that, as each neighborhood of a material point is allowed to distort (*i.e.* to change its relaxed state) without constraint, the integrity of the body may not be preserved (compenetration, cavitation), unless an elastic deformation $\mathbf{F}_e$ takes place to accommodate the distortion $\mathbf{F}_o$, and realizes the final visible deformation $\mathbf{F}$; thus, the (right Cauchy–Green) strain tensor $\mathbf{C}=\mathbf{F}^T\mathbf{F}$ is the metric tensor of the realized configuration $\chi(\mathcal{B})$. We assume the elastic strain energy density φ associated to the mechanical state $(\chi,\mathbf{F}_o)$ to be a convex function $\hat{\varphi}$ of the *elastic strain measure* $\mathbf{C}_e=\mathbf{F}_e^T\mathbf{F}_e$