Journal of the Mechanics and Physics of Solids 57 (2009) 1449–1458

Contents lists available at ScienceDirect

Journal of the Mechanics and Physics of Solids

journal homepage: www.elsevier.com/locate/jmps

The influence of particle fluctuations on the average rotation in an idealized granular material

Luigi La Ragione [a,*], James T. Jenkins [b]

[a] Dipartimento di Ingegneria Civile e Ambientale, Politecnico di Bari, 70125 Bari, Italy
[b] Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, NY 14853, USA

ARTICLE INFO

Article history:
Received 13 November 2008
Received in revised form
1 March 2009
Accepted 12 June 2009

Keywords:
A. Effective moduli
B. Granular material
Cosserat material
C. Probability and statistics
Numerical simulation

ABSTRACT

Granular materials are a simple example of a Cosserat continuum in that the average particle rotations may differ from the rotation of the average deformation. In the absence of couple stress, this difference insures that the stress is symmetric. This has been shown in theories that assume that the displacement at particle contacts is given by the average deformation and spin. Here, we indicate how the difference between the average rotation of the particles and the average rotation of the deformation can be determined when fluctuations in particle displacements and rotations satisfy local force and moment equilibria in a random aggregate of identical spheres. The predictions based on this model are in better agreement with numerical simulation than that given by the simple average strain theory.

1. Introduction

The increment in the average particle rotation of an idealized granular material consisting of a random aggregate of identical frictional spheres may be determined from micromechanical considerations. In earlier work (Koenders, 1990; Jenkins, 1991; Jenkins and La Ragione, 2001), it was assumed that the contact deformation was directly related to the average deformation (strain and rotation). This was used with an expression for the contact force between the particles and a simple average over all contacts in the aggregate to derive an expression for the stress tensor. Then, by requiring that the stress be symmetric, an expression of the average rotation of the particles was obtained. However, more physically realistic kinematic descriptions require that fluctuations in displacements and rotations also be incorporated (Koenders, 1987, 1994; Chang and Liao, 1990). When such fluctuations are included, the particles can be equilibrated. Then, realistic elastic moduli for random aggregates of spheres (La Ragione and Jenkins, 2007) and disks (Jenkins and Koenders, 2004) can be predicted.

Here, we incorporate fluctuations and show how their presence influences the way in which the average rotation of the particles is determined. Chang and Liao (1990), Misra and Chang (1993) also introduce fluctuations to derive a stress–strain relation for a random aggregate of elastic–frictional spheres, but only paid particular attention to the average particle rotations when couple stresses were present. Koenders (1994) also incorporated fluctuations in a similar study, but neglected them in his determination of the average particle rotations from a weighted sum of the moment equilibrium equations.

We consider homogeneous quasi-static deformation in which the couple stress is identically zero and introduce incremental fluctuations in translation and rotation that satisfy force and moment equilibrium. Because moment

* Corresponding author. Tel.: +39 80 596 3739.
E-mail address: l.laragione@poliba.it (L. La Ragione).

0022-5096/$ - see front matter
doi:10.1016/j.jmps.2009.06.005

equilibrium is satisfied for each particle, the weighted sum of these is also satisfied and does not serve to determine the difference in the incremental rotations. Instead, the difference in the increments of the average rotations is determined by the requirement that the fluctuations in the average particle rotations vanish. In general, this results in a different value of the increment of the average particle rotations than has been obtained previously.

2. Kinematics

We characterize the incremental response of an aggregate of N identical elastic spheres with diameter d that, after an initial isotropic compression, has been sheared. We phrase the problem incrementally in order to make use of the resulting linear relations between the increments in the force and displacement in the equations of equilibrium. We employ objective increments of quantities, to insure that they transform as tensors under time-dependent changes of frame, because our main interest is in predicting the average rotation of the particles.

The incremental relative displacement $\dot{\mathbf{u}}^{(BA)}$ of the point of contact of particle B relative to that of particle A is

$$\dot{u}_i^{(BA)} = \dot{c}_i^{(B)} - \dot{c}_i^{(A)} - \tfrac{1}{2}\varepsilon_{ijk}(\dot{\omega}_j^{(B)} + \dot{\omega}_j^{(A)})d_k^{(BA)}, \tag{1}$$

where, for example, $\dot{\mathbf{c}}^{(B)}$ and $\dot{\boldsymbol{\omega}}^{(B)}$ are, respectively, increments in the translation of the center and the rotation of particle B, and $\mathbf{d}^{(BA)}$ is the vector from the center of particle A to the center of B. The translations and rotations can be considered as sums of an average and a fluctuation. Therefore, we write the increment of the fluctuation in the difference between the translations of the two centers as

$$\dot{c}_i^{(B)} - \dot{c}_i^{(A)} \equiv (D_{ij} + W_{ij})d_j^{(BA)} + \dot{c}_i^{(B)'} - \dot{c}_i^{(A)'}, \tag{2}$$

where $\mathbf{D}$ and $\mathbf{W}$ are, respectively, the symmetric and antisymmetric parts of the increment in the average deformation gradient and the prime denotes a fluctuation. In a similar way, the increment in the fluctuation in the sum of the rotations of the two contacting particles is

$$\dot{\omega}_i^{(B)} + \dot{\omega}_i^{(A)} \equiv 2\Omega_i^{\times} + \dot{\omega}_i^{(B)'} + \dot{\omega}_i^{(A)'}, \tag{3}$$

where $\boldsymbol{\Omega}^{\times}$ is the axial vector of the skew-symmetric tensor $\boldsymbol{\Omega}$, the rate of the average rotation. Here, the averages $\mathbf{D}$ and $\mathbf{W}$ are regarded as known, and $\boldsymbol{\Omega}$ is to be determined.

We introduce the definitions $\dot{\boldsymbol{\Delta}}^{(BA)} \equiv \dot{\mathbf{c}}^{(B)'} - \dot{\mathbf{c}}^{(A)'}$ and $\dot{\mathbf{S}}^{(BA)} \equiv \dot{\boldsymbol{\omega}}^{(B)'} + \dot{\boldsymbol{\omega}}^{(A)'}$ and the objective co-rotational rates

$$\overset{*}{u}{}_i^{(BA)} \equiv \dot{u}_i^{(BA)} - W_{ij}u_j^{(BA)} \tag{4}$$

and

$$\overset{*}{\Delta}{}_i^{(BA)} \equiv \dot{\Delta}_i^{(BA)} - W_{ij}\Delta_j^{(BA)}. \tag{5}$$

Then, assuming that the structure of the aggregate does not change during deformation, Eq. (1) may be written

$$\overset{*}{u}{}_i^{(BA)} = (D_{ij} + W_{ij})d_j^{(BA)} + \overset{*}{\Delta}{}_i^{(BA)} - \tfrac{1}{2}\varepsilon_{ijk}(2\Omega_j^{\times} + \dot{S}_j^{(BA)})d_k^{(BA)}. \tag{6}$$

We note that the difference in the average rates of rotations and $\dot{\mathbf{S}}^{(BA)}$ are objective quantities, while the average rates of rotation taken separately are not.

3. Local equilibrium

The relation between the incremental contact force exerted by B on A and the incremental displacement of B relative to A is

$$\dot{F}_i^{(BA)} = K_{ij}^{(BA)}\dot{u}_i^{(BA)} \tag{7}$$

or, equivalently,

$$\overset{*}{F}{}_i^{(BA)} = K_{ij}^{(BA)}\,\overset{*}{u}{}_i^{(BA)}, \tag{8}$$

where K_{ij} is the contact stiffness tensor. The response of the contact may be elastic or it may involve frictional sliding. When sliding occurs, the tangential stiffness vanishes.

We require that force and moment equilibrium be satisfied for each particle. Then, for example, for particle A,

$$\sum_{n=1}^{N^{(A)}} \overset{*}{F}{}_i^{(nA)} = \sum_{n=1}^{N^{(A)}} \dot{F}_i^{(nA)} + W_{ij}\sum_{n=1}^{N^{(A)}} F_j^{(nA)} = \sum_{n=1}^{N^{(A)}} \dot{F}_i^{(nA)} = 0 \tag{9}$$