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A variational model for fracture mechanics: Numerical experiments

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Abstract

In the variational model for brittle fracture proposed in Francfort and Marigo [1998. Revisiting brittle fracture as an energy minimization problem. J. Mech. Phys. Solids 46, 1319–1342], the minimum problem is formulated as a free discontinuity problem for the energy functional of a linear elastic body. A family of approximating regularized problems is then defined, each of which can be solved numerically by a finite element procedure. Here we re-formulate the minimum problem within the context of finite elasticity. The main change is the introduction of the dependence of the strain energy density on the determinant of the deformation gradient. This change requires new, more general existence and Γ-convergence results. The results of some two-dimensional numerical simulations are presented, and compared with corresponding simulations made in Bourdin et al. [2000. Numerical experiments in revisited brittle fracture. J. Mech. Phys. Solids 48, 797–826] for the linear elastic model.

Keywords: Fracture mechanics; Energy methods; Variational calculus; Free-boundary problems; Finite elements

1. Introduction

A variational formulation for the evolution of the fracture surface in a brittle, linearly elastic solid was given by Francfort and Marigo (1998). In Bourdin et al. (2000), the same

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formulation was used in the numerical solution of some two-dimensional model problems. The aim of the present paper is to extend the results of Francfort and Marigo (1998) and Bourdin et al. (2000) to the context of fully non-linear elasticity. Crucial in our reformulation is the assumption of the dependence of the strain energy density on the determinant of the deformation gradient. In fact, this assumption provides a more realistic picture of the fracturing process, though at the price of some technical difficulties.

The energy functional assumed in Bourdin et al. (2000) is the sum of two terms, a bulk energy depending quadratically on the symmetric part of the gradient of the displacement vector, and a surface energy proportional to the area of the fracture surface. The first is the strain energy of a linear elastic body, and the second is the fracture energy in Griffith's theory for brittle fracture (Griffith, 1920). In the case of antiplane elasticity the displacement vector reduces to a scalar, and the energy functional becomes similar to the functional of Mumford and Shah in the problem of image segmentation (Mumford and Shah, 1989). Then one may borrow from this problem the existence theory, based on Ambrosio's theorems of compactness and lower semicontinuity in the set SBV of *special* functions of bounded variation (Ambrosio, 1994; De Giorgi and Ambrosio, 1988).

One may also use a numerical solution technique, based on a theorem of Ambrosio and Tortorelli (1990, 1992), for the approximation, in the sense of Γ-convergence, of the Mumford–Shah functional by means of regularized functionals. For fracture problems, the regularized functionals are defined in Sobolev spaces, and the finite element method of classical structural analysis can be used. In Bourdin et al. (2000) this technique was successfully applied to problems of antiplane shear. The same technique was also used to solve problems of plane elasticity; this extrapolation to the vectorial case was fully legitimated later, by a Γ-convergence result proved by Chambolle (2004).

For the case of fully non-linear elasticity considered in the present paper, an appropriate extension of the mathematical background of Bourdin et al. (2000) is required. This was partially done in Ambrosio and Braides (1995), where the existence of minimizers for polyconvex functionals was proved using a lower semicontinuity theorem in SBV due to Ambrosio (1994). However, the growth condition assumed in the theorem, see Eq. (6) below, is not obeyed by the bulk energy of a significant class of materials, including the Ogden materials which we consider in our analysis. This obstacle can be removed using a different lower semicontinuity result for polyconvex functionals, recently proved by Fusco et al. (2006), see the discussion in Section 3. This result can be viewed as the SBV counterpart of Ball's (1977) lower semicontinuity theorem in Sobolev spaces.

Moreover, a formulation within the context of non-linear elasticity requires a vectorial counterpart of Ambrosio and Tortorelli's Γ-convergence theorem. In this respect, only a partial result is available so far. It can be obtained by combining a Γ-convergence theorem proved by Focardi (2001) in the vectorial setting, with the lower semicontinuity theorem of Fusco et al. (2006) for polyconvex functionals, see the discussion in Section 4. Hence, a rigorous justification of the numerical technique used in this paper requires further mathematical work.

The numerical solution technique, which is the same used in Bourdin et al. (2000), is also far from being rigorous. Indeed, the solution algorithm is based on the alternate minimization with respect to two independent variables, and for it no convergence proof is available. Even worse, in our extension to the non-linear case, due to the non-convexity of the bulk energy density, each partial minimization is replaced by the determination of a stationary point.