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ScienceDirect
Journal of the Mechanics and Physics of Solids 55 (2007) 1993–2006
JOURNAL OF THE MECHANICS AND PHYSICS OF SOLIDS
www.elsevier.com/locate/jmps

Three-dimensional solutions for general anisotropy

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Received 21 November 2006; received in revised form 1 February 2007; accepted 6 February 2007

Abstract

The Stroh formalism is extended to provide a new class of three-dimensional solutions for the generally anisotropic elastic material that have polynomial dependence on x_3, but which have quite general form in x_1, x_2. The solutions are obtained by a sequence of partial integrations with respect to x_3, starting from Stroh's two-dimensional solution. At each stage, certain special functions have to be introduced in order to satisfy the equilibrium equation. The method provides a general analytical technique for the solution of the problem of the prismatic bar with tractions or displacements prescribed on its lateral surfaces. It also provides a particularly efficient solution for three-dimensional boundary-value problems for the half-space. The method is illustrated by the example of a half-space loaded by a linearly varying line force.

Keywords: Anisotropic elasticity; Stroh formalism; Three-dimensional problems; Hierarchical methods; Boundary-value problems

1. Introduction

The constitutive law for a generally anisotropic material involves 21 independent elastic constants. Methods are well established for the solution of such problems for cases where the stress and displacement fields depend on only two of the three spatial coordinates x_1, x_2, x_3. Lekhnitskii (1963) starts from expressions for the stresses in terms of stress functions that satisfy equilibrium and shows that the compatibility condition can then be

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0022-5096/$ - see front matter
doi:10.1016/j.jmps.2007.02.002

decomposed into six first order operators. Alternatively, Stroh (1958, 1962) shows that particular solutions can be found in which the displacement vector has the same direction at all points and a magnitude that is a function of a certain complex combination of x_1, x_2. The general solution is then written as a sum of these solutions. (For a detailed exposition of Stroh's solution including numerous examples, see Ting, 1996.) Both the Stroh and Lekhnitskii methods can be regarded as emanating from appropriate linear transformations of the in-plane coordinates x_1, x_2 and require that these transformations be distinct. For example, if two of Lekhnitskii's first order operators or two of Stroh's combinations of x_1, x_2 should be identical, special methods are necessary for the solution. This condition arises only for certain special combinations of elastic constants (including of course the case of isotropy) and will not be considered in the present paper.

Very few solutions exist for problems of general anisotropy when the stresses depend on all three coordinates. Published solutions, such as those for a concentrated point force or dislocation in an infinite body or a concentrated force on the surface of a half-space are generally obtained using transform methods, such that the problem in the transform domain is two-dimensional and can therefore be treated using the Stroh or Lekhnitskii formalism (Sveklo, 1964; Willis, 1966; Ting, 2006; Wu, 1998). This contrasts with isotropic elasticity, where (for example) general solutions can be expressed in terms of three-dimensional harmonic functions using the Papkovitch–Neuber formulation (Barber, 2002). Barber (2006a) has shown how certain three-dimensional isotropic solutions can be derived from their two-dimensional counterparts by successive partial integrations in the x_3-direction. This leads to a general solution of the problem of an isotropic prismatic bar loaded on its lateral surfaces, provided only that these loads can be expressed as finite power series in x_3. At each stage in the integration process, a two-dimensional problem is solved to ensure that the lowest order terms in the solution satisfy: (i) the equations of elasticity and (ii) the boundary conditions. The hierarchical structure underlying this procedure was first enunciated by Ieşan (1986) and has also been applied by other authors in both analytical and numerical formulations (Ladevèze et al., 2004; Huang and Dong, 2001).

An essentially similar procedure can be applied to problems in general anisotropy (Rand and Rovenski, 2005). However, in the isotropic case, the equations of elasticity can be reduced to the condition that the Papkovitch–Neuber potentials satisfy Laplace's equation and the development of partial integrals satisfying this condition is a routine problem in potential theory. For the anisotropic case, this strategy is no longer available, except for certain special cases such as that of transverse isotropy.

The problem of determining an appropriate partial integral can be reduced to a sequence of two-dimensional body force problems (Barber, 2006b), which in turn could be solved by convolution on the known line force solution. However, this procedure is extremely cumbersome in practice. In the present paper, we shall extend the classical Stroh formalism for two-dimensional general anisotropy to stress and displacement fields with polynomial dependence on the third coordinate x_3. In particular, we shall show that the general solution for a stress field with polynomial dependence on x_3 can be written as a series involving powers of x_2, x_3 multiplying arbitrary functions of $x_1 + px_2$, where p is one of the Stroh eigenvalues and we shall develop a set of recurrence relations for determining the coefficients in this series that depend only on the elastic constants and not on the particular boundary-value problem. The method is illustrated with the problem of a half-space loaded by a linearly-varying line load.