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Steady accretion of an elastic body on a hard spherical surface and the notion of a four-dimensional reference space

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ABSTRACT

Taking the cue from experiments on actin growth on spherical beads, we formulate and solve a model problem describing the accretion of an incompressible elastic solid on a rigid sphere due to attachment of diffusing free particles. One of the peculiar characteristics of this problem is that accretion takes place on the interior surface that separates the body from its support rather than on its exterior surface, and hence is responsible for stress accumulation. Simultaneously, ablation takes place at the outer surface where material is removed from the body. As the body grows, mechanical effects associated with the build-up of stress and strain energy slow down accretion and promote ablation. Eventually, the system reaches a point where internal accretion is balanced by external ablation. The present study is concerned with this stationary regime called "treadmilling".

The principal ingredients of our model are: a nonstandard choice of the reference configuration, which allows us to cope with the continually evolving material structure; and a driving force and a kinetic law for accretion/ablation that involves the difference in chemical potential, strain energy and the radial stress. By combining these ingredients we arrive at an algebraic system which governs the stationary treadmilling state. We establish the conditions under which this system has a solution and we show that this solution is unique. Moreover, by an asymptotic analysis we show that for small beads the thickness of the solid is proportional to the radius of the support and is strongly affected by the stiffness of the solid, whereas for large beads the stiffness of the solid is essentially irrelevant, the thickness being proportional to a characteristic length that depends on the parameters that govern diffusion and accretion kinetics.

1. Introduction

Surface growth, i.e. the accretion of a solid onto a surface, occurs in several contexts of physical, technological, and biological interest. One of the most common examples of surface growth is the solidification of water at the ice–water interface near the freezing temperature; other examples include technological processes such as chemical vapor deposition or, in biology, the growth of hard tissues like bones and teeth.

Although surface growth may be regarded as bulk growth concentrated on a surface (DiCarlo, 2005), surface and bulk growth are in general treated in a different manner. When dealing with bulk growth, the reference configuration is fixed and

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the addition of particles to the body is accounted for by a tensor field, often referred to as the *growth tensor* (Rodriguez et al., 1994), whose value at a given point identifies the stress-free stance (DiCarlo and Quiligotti, 2002) of a chunk of material in a small neighborhood of that point. When dealing with surface growth, on the other hand, it seems more natural to account for addition and removal of material by letting the boundary of the reference configuration evolve, as done in Skalak et al. (1997).

The choice of a reference configuration in bulk growth is rarely an issue in fact, because of the extra degree of freedom brought in by the growth tensor, there is always a pair of a deformation and of a growth-tensor field that identifies a stress-free state. For surface growth, on the contrary, it is not always obvious what is the most convenient choice of a reference configuration. In particular, it might be impossible to identify a stress-free state through the conventional notion of reference configuration in a three-dimensional reference space. From the kinematic standpoint, a resolution to this difficulty was provided by Skalak et al. (1982) in a seminal paper by introducing the time τ at which a material point is deposited on the growth surface. They label each particle of the body at time t with four coordinates (a_1, a_2, τ, t), where (a_1, a_2) are the coordinates of the point on the two-dimensional growth surface at which the material point was deposited. It is this basic idea that we build upon in our developments.

Another important feature of surface growth is the dependence of the accretion rate on the local stress field. Often, accretion happens at the outer surface of the body, with each new layer of material forming on top of the layer that was last formed. If each new layer is geometrically compatible with the previous one, accretion does not lead to a build-up of stress. On the other hand if accretion occurs on an interior surface of the body, and each new layer of material has to push away the previous layer, then this necessarily generates a residual stress in the body. The stress field, by its turn, appears in the laws governing accretion rate through an Eshelbian-like coupling term (Ambrosi and Guana, 2007; Ciarletta et al., 2013; DiCarlo and Quiligotti, 2002; Epstein and Maugin, 2000). All these effects result into an intimate coupling between mechanics and growth.

The specific problem we consider is described schematically in Fig. 1. A note on terminology first, since we wish to refer to the individual units of material that combine to form a body, it is convenient to refer to any such unit as a “free particle”; to the body formed by the combination of many free particles as the “solid”; to the processes of adding and removing free particles from the solid as “accretion” and “ablation”, respectively. In Fig. 1 the solid body $\mathcal{B}$ occupies the region between the two spherical surfaces Σ_0 and $\Sigma_1(t)$. The inner surface Σ_0 is the boundary of a rigid bead on whose surface accretion occurs. The bead is surrounded by a fluid in which the free particles are dispersed, the fluid occupying the entire region outside Σ_0, wherefore it both surrounds and permeates the solid. The free particles diffuse towards the bead surface due to a gradient in chemical potential. When they arrive at Σ_0 they attach to $\mathcal{B}$. At the same time, it turns out that it is energetically favorable for the solid to shed free particles at its outer surface $\Sigma_1(t)$, wherefore ablation occurs simultaneously at $\Sigma_1(t)$. Thus free particles are continually being attached and detached from the body, the former at the inner surface and the latter at the outer surface. If the rate of accretion is greater than the rate of ablation, the body grows and $\Sigma_1(t)$ moves outwards. The evolving stress and deformation fields within the solid are governed by a mechanics problem. The flow of free particles is governed by a diffusion problem. And these two problems are coupled at the two surfaces of the body through both the conservation of free particles and the kinetics of accretion (when there is no ambiguity, as in the preceding sentence, we

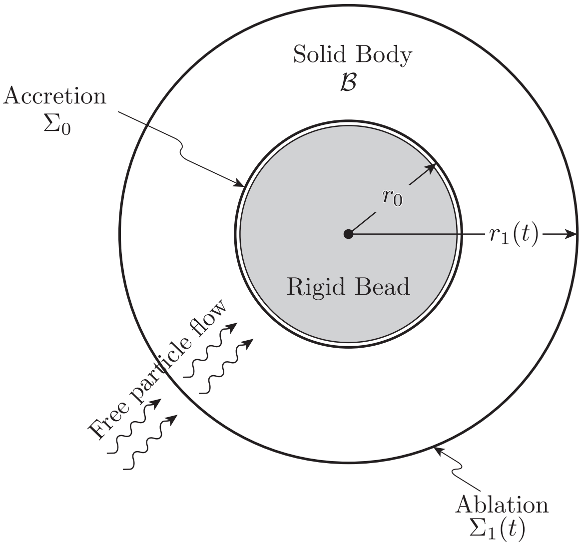

Fig. 1. Problem setting: an annular solid body $\mathcal{B}$, bounded by two concentric spherical surfaces Σ_0 and $\Sigma_1(t)$. The former represents the fixed support at which accretion occurs, the later is the (in general) time-dependent outer surface of the body. A fluid containing “free particles” fills the entire region outside Σ_0, and therefore surrounds and permeates the solid. The free particles diffuse through the solid to reach its inner surface Σ_0 where they attach to the solid. Free particles are continually being attached to the body at the inner surface and detached from the body at the outer surface.