Journal of the Mechanics and Physics of Solids 60 (2012) 261–276

Contents lists available at SciVerse ScienceDirect

Journal of the Mechanics and Physics of Solids

journal homepage: www.elsevier.com/locate/jmps

Strain gradient solution for the Eshelby-type polyhedral inclusion problem

X.-L. Gao [a,]*, M.Q. Liu [b]

[a] Department of Mechanical Engineering, University of Texas at Dallas, 800 West Campbell Road, Richardson, TX 75080-3021, USA
[b] Department of Mechanical Engineering, Texas A&M University, College Station, TX 77843-3123, USA

A R T I C L E I N F O

Article history:
Received 22 July 2011
Received in revised form
27 October 2011
Accepted 28 October 2011
Available online 6 November 2011

Keywords:
Eshelby tensor
Polyhedral inclusion
Size effect
Eigenstrain
Strain gradient

A B S T R A C T

The Eshelby-type problem of an arbitrary-shape polyhedral inclusion embedded in an infinite homogeneous isotropic elastic material is analytically solved using a simplified strain gradient elasticity theory (SSGET) that contains a material length scale parameter. The Eshelby tensor for a polyhedral inclusion of arbitrary shape is obtained in a general analytical form in terms of three potential functions, two of which are the same as the ones involved in the counterpart Eshelby tensor based on classical elasticity. These potential functions, as volume integrals over the polyhedral inclusion, are evaluated by dividing the polyhedral inclusion domain into tetrahedral duplexes, with each duplex and the associated local coordinate system constructed using a procedure similar to that employed by Rodin (1996. J. Mech. Phys. Solids 44, 1977–1995). Each of the three volume integrals is first transformed to a surface integral by applying the divergence theorem, which is then transformed to a contour (line) integral based on Stokes' theorem and using an inverse approach different from those adopted in the existing studies based on classical elasticity. The newly derived SSGET-based Eshelby tensor is separated into a classical part and a gradient part. The former contains Poisson's ratio only, while the latter includes the material length scale parameter additionally, thereby enabling the interpretation of the inclusion size effect. This SSGET-based Eshelby tensor reduces to that based on classical elasticity when the strain gradient effect is not considered. For homogenization applications, the volume average of the new Eshelby tensor over the polyhedral inclusion is also provided in a general form. To illustrate the newly obtained Eshelby tensor and its volume average, three types of polyhedral inclusions – cubic, octahedral and tetrakaidecahedral – are quantitatively studied by directly using the general formulas derived. The numerical results show that the components of the SSGET-based Eshelby tensor for each of the three inclusion shapes vary with both the position and the inclusion size, while their counterparts based on classical elasticity only change with the position. It is found that when the inclusion is small, the contribution of the gradient part is significantly large and should not be neglected. It is also observed that the components of the averaged Eshelby tensor based on the SSGET change with the inclusion size: the smaller the inclusion, the smaller the components. When the inclusion size becomes sufficiently large, these components are seen to approach (from below) the values of their classical elasticity-based counterparts, which are constants independent of the inclusion size.

* Corresponding author. Tel.: +1 972 883 4550; fax: +1 972 883 4659.
E-mail address: Xin-Lin.Gao@utdallas.edu (X.-L. Gao).

0022-5096/$ - see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jmps.2011.10.010

1. Introduction

Eshelby's (1957, 1959) solution for the problem of an infinite homogeneous isotropic elastic material containing an ellipsoidal inclusion prescribed with a uniform eigenstrain is a milestone in micromechanics. The solution for the dynamic Eshelby ellipsoidal inclusion problem was obtained by Michelitsch et al. (2003), which reduces to the Eshelby solution in the static limiting case. Both of these solutions are based on classical elasticity. Recently, the Eshelby ellipsoidal inclusion problem was solved by Gao and Ma (2010b) using a simplified strain gradient elasticity theory, which recovers Eshelby's (1957, 1959) solution when the strain gradient effect is not considered.

A remarkable property of Eshelby's (1957) solution is that the Eshelby tensor, which is a fourth-order strain transformation tensor directly linking the induced strain to the prescribed uniform eigenstrain, is constant inside the inclusion. However, this property is true only for ellipsoidal inclusions (and when classical elasticity is used), which is known as the Eshelby conjecture (e.g., Eshelby, 1961; Rodin, 1996; Markenscoff, 1998a,b; Lubarda and Markenscoff, 1998; Liu, 2008; Li and Wang, 2008; Gao and Ma, 2010b; Ammari et al., 2010).

For non-ellipsoidal polyhedral inclusions, Rodin (1996) provided an algorithmic analytical solution and showed that Eshelby's tensor cannot be constant inside a polyhedral inclusion, thereby proving the Eshelby conjecture in the case of polyhedral inclusions. The expressions of Eshelby's tensor for two-dimensional (2-D) polygonal inclusions were included in Rodin (1996). The explicit expressions of the Eshelby tensor for three-dimensional (3-D) polyhedral inclusions were later derived by Nozaki and Taya (2001), where an exact solution for the stress field inside and outside an arbitrary-shape polyhedral inclusion was obtained and numerical results for five regular polyhedral inclusion shapes and three other shapes of the icosidodeca family were presented. Both Rodin (1996) and Nozaki and Taya (2001) made use of an algorithm developed by Waldvogel (1979) for evaluating the Newtonian (harmonic) potential over a polyhedral body. A more compact form of the Eshelby tensor than that presented in Nozaki and Taya (2001) for a polyhedral inclusion in an infinite elastic space was proposed by Kuvshinov (2008) using a coordinate-invariant formulation, where problems of polyhedral inclusions in an elastic half-space and bi-materials were also investigated. In addition, specific analytical solutions have been obtained for polyhedral inclusions of simple shapes such as cuboids (e.g., Chiu, 1977; Lee and Johnson, 1978; Liu and Wang, 2005) and pyramids (e.g., Pearson and Faux, 2000; Glas, 2001; Nenashev and Dvurechenskil, 2010). Also, illustrative results have been provided for dynamic Eshelby problems of cubic and triangularly prismatic inclusions along with spherical and ellipsoidal ones by Wang et al. (2005) using their general solution for the dynamic Eshelby problem for inclusions of various shapes.

However, these existing studies on polyhedral inclusion problems are all based on the classical elasticity theory, which does not contain any material length scale parameter. As a result, the Eshelby tensors obtained in these studies and the subsequent homogenization methods cannot capture the inclusion (particle) size effect on elastic properties exhibited by particle–matrix composites (e.g., Vollenberg and Heikens, 1989; Cho et al., 2006; Marcadon et al., 2007). Solutions for polyhedral inclusion problems are also important for describing interpenetrating phase composites reinforced by 3-D networks (e.g., Poniznik et al., 2008; Jhaver and Tippur, 2009) and for understanding semiconductor materials buried with quantum dots that are typically polyhedral-shaped (e.g., Kuvshinov, 2008; Nenashev and Dvurechenskil, 2010). These materials often exhibit microstructure-dependent size effects whose interpretation requires the use of higher-order continuum theories.

In this paper, the Eshelby-type inclusion problem of a polyhedral inclusion prescribed with a uniform eigenstrain and a uniform eigenstrain gradient and embedded in an infinite homogeneous isotropic elastic material is solved using a simplified strain gradient elasticity theory (SSGET) (e.g., Gao and Park, 2007), which contains a material length scale parameter and can describe size-dependent elastic deformations. The Eshelby tensor is analytically obtained in terms of three potential functions, two of which are the same as the ones involved in the counterpart Eshelby tensor based on classical elasticity. These potential functions, as three volume integrals over the polyhedral inclusion, are evaluated by dividing the polyhedral inclusion domain into tetrahedral duplexes. Each duplex and the associated local coordinate system are constructed using a procedure similar to that developed by Rodin (1996) based on the algorithm proposed in Waldvogel (1979). Each of the three volume integrals is first transformed to a surface integral by applying the divergence theorem, which is then transformed to a contour (line) integral based on Stokes' theorem and using an inverse approach different from those employed in the existing studies for evaluating the two integrals involved in the classical elasticity-based Eshelby tensor for a polyhedral inclusion.

The rest of this paper is organized as follows. In Section 2, the general form of the Eshelby tensor for a 3-D arbitrary-shape inclusion based on the SSGET is presented in terms of three potential functions (volume integrals). The expressions of the SSGET-based Eshelby tensor for a polyhedral inclusion of arbitrary shape are analytically derived in Section 3, which is separated into a classical part and a gradient part. The averaged Eshelby tensor over the inclusion volume is also analytically evaluated there. Numerical results are provided in Section 4 to quantitatively illustrate the position and inclusion size dependence of the newly obtained Eshelby tensor for the polyhedral inclusion problem. The paper concludes in Section 5.

2. Eshelby tensor based on the SSGET

The SSGET is the simplest strain gradient elasticity theory evolving from Mindlin's pioneering work. It is also known as the first gradient elasticity theory of Helmholtz type and the dipolar gradient elasticity theory (e.g., Gao and Ma, 2010a).