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Travelling wave solutions for a quasilinear model of field dislocation mechanics

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ABSTRACT

We consider an exact reduction of a model of Field Dislocation Mechanics to a scalar problem in one spatial dimension and investigate the existence of static and slow, rigidly moving single or collections of planar screw dislocation walls in this setting. Two classes of drag coefficient functions are considered, namely those with linear growth near the origin and those with constant or more generally sublinear growth there. A mathematical characterisation of all possible equilibria of these screw wall microstructures is given. We also prove the existence of travelling wave solutions for linear drag coefficient functions at low wave speeds and rule out the existence of nonconstant bounded travelling wave solutions for sublinear drag coefficients functions. It turns out that the appropriate concept of a solution in this scalar case is that of a viscosity solution. The governing equation in the static case is not proper and it is shown that no comparison principle holds. The findings indicate a short-range nature of the stress field of the individual dislocation walls, which indicates that the nonlinearity present in the model may have a stabilising effect. We predict idealised dislocation-free cells of almost arbitrary size interspersed with dipolar dislocation wall microstructures as admissible equilibria of our model, a feature in sharp contrast with predictions of the possible non-monotone equilibria of the corresponding Ginzburg–Landau phase field type gradient flow model.

1. Introduction

The objective of this paper is to deduce some mathematically rigorous results related to solutions of the theory of field dislocation mechanics (FDM), see Acharya (2004, 2010). FDM is a nonlinear, dynamical, dissipative PDE model of dislocation mechanics that aims to treat single and collections of dislocation defects as nonsingular localisations of a dislocation density field. It includes elastic nonconvexity to reflect lattice periodicity and predicts dislocation internal stress and permanent deformation due to dislocation motion. Here, we prove existence of solutions to a special, but exact, class of problems within FDM and characterise the entire class of bounded equilibria and travelling wave solutions of this class for low wave speeds.

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Physically, the solutions we explore represent static and rigidly moving single or collections of planar screw dislocation walls, perpendicular to the axis of an at most homogeneously sheared cylinder. Any given wall consists of screw dislocations of the same sign, but two walls may be of different sign in this sense. A particular result is the characterisation of all equilibria of such walls under no applied deformation; i.e., the class of residually stressed, static dislocation microstructure consisting of screw dislocation walls. Walls of screw dislocations are important microstructural features that have found practical application, e.g., in epitaxial growth (Matthews, 1974) and enhancement of ductility (Wunderlich et al., 1993). Zero stress walls are discussed in Head et al. (1993), Roy and Acharya (2005), and Limkumnerd and Sethna (2007).

Mathematically, we characterise all possible bounded equilibria of these screw wall microstructures, for two classes of drag coefficient functions, namely those with linear growth near the origin and those with sublinear growth there. We also prove the existence of travelling wave solutions for linear drag coefficient function at low wave speeds and rule out the existence of nonconstant bounded travelling wave solutions for sublinear drag coefficient functions. The governing equation is quasilinear (see (1)); it becomes degenerate if the quotient F of ϕ_x^2 and the drag coefficient function vanishes. It is this degeneracy that leads to a plethora of solutions for the equilibrium equation and the dynamic (travelling wave) equation for drag coefficient functions with linear growth. In essence, it becomes possible to glue together certain solution segments, as discussed below, to obtain new solutions. This intuitive approach can be made rigorous with a suitable variant of the notion of viscosity solutions, defined in Appendix A. The notion is weaker than that of the more classical case of proper equations, and we show that for the equation under consideration, no comparison principle holds. (Viscosity solutions were first developed for Hamilton Jacobi equations, where an interpretation based on viscous regularisations can be made; in the context considered here, viscosity solutions are not related to physical viscosity).

The equation under consideration is related to the van der Waals energy and its gradient flow. There is an enormous body of beautiful results on this subject, which we cannot attempt to survey appropriately, so we just mention a few papers by Carr et al. (1984), Fife and McLeod (1980/81), Carr and Pego (1989) and Bronsard and Kohn (1991). A key difference between the analysis of equations of the type $\phi_t = \varepsilon^2\phi_{xx} - f(\phi)$ and the equation considered here is the degeneracy of our equation (see (1)), which brings with it a multitude of equilibria (and travelling waves) and requires us to consider a suitable concept of viscosity solutions. For studies of attractors of scalar nondegenerate parabolic equations, we refer the reader to Fiedler and Rocha (1996) and Härterich (1998). We also mention related work by Alber and Zhu (2005) on a model for martensitic phase transitions which involves a degenerate parabolic equation which resembles the equation studied here for the constant drag coefficient function. Alber and Zhu (2005) prove the existence to an initial value problem by introducing a regularisation of the term responsible for the degeneracy and considering the limit of vanishing regularisation.

Walls of singular screw dislocations in the linear elastic context are a somewhat frequently discussed topic; some representative samples are Li and Needham (1960), Hovakimian and Tanaka (1998), Roy et al. (2008) and of course the classic book by Hirth and Lothe (1982). To our knowledge, there is no prior work that considers mathematically rigorous analysis of a dynamic model of dislocations with elastic nonconvexity. The analysis that comes closest to our considerations is that of Carpio et al. (2001) but they consider dislocation configurations that do not interact through their stress fields.

The paper is organised as follows. Section 2 contains a brief description of the PDE model we consider. Section 3 characterises all bounded equilibrium solutions, both for linear and superlinear F (that is, linear and sublinear drag coefficient function). Section 4 studies travelling wave solutions for the model discussed in this paper. Some of the equilibrium and the travelling wave solutions have to be interpreted in the sense of viscosity solutions. To make the article self-contained, a definition of viscosity solutions is given in Appendix A. A brief discussion is the content of Section 6.

2. Governing equation for the dynamics of screw dislocation walls

We consider an infinite cylinder, of rectangular cross section for definiteness, containing walls of screw dislocations. The bottom of the cylinder is held fixed and the cylinder is sheared on the top surface by an applied displacement boundary condition along the horizontal in-plane direction. The applied displacement is spatially uniform on the top surface. These facts are described schematically in Fig. 1. We describe briefly the elements of an ansatz leading to an exact problem in one spatial dimension; details of the derivation can be found in Acharya (2010).

All fields are assumed to be uniform in y and z and therefore can be thought of mathematically as being only functions of x and t, where t is time. Further, u represents the displacement on the top surface of the cylinder in the y direction, and g represents the yz component of the total shear distortion. The only non-zero plastic distortion component is ϕ, which

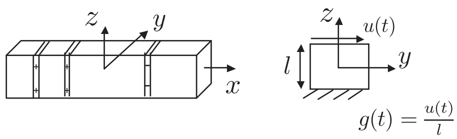

Fig. 1. Coordinates in the infinite cylinder under consideration.