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A variational formulation of the coupled thermo-mechanical boundary-value problem for general dissipative solids

Q. Yang[a], L. Stainier[b], M. Ortiz[a,*]

[a]Division of Engineering and Applied Science, California Institute of Technology, Pasadena, CA 91125-5000, USA
[b]Department of Aerospace, Mechanics and Materials, Université de Liège, B-4000 Liège, Belgium

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Abstract

A variational formulation of the coupled thermo-mechanical boundary-value problem for general dissipative solids is presented. The coupled thermo-mechanical boundary-value problem under consideration consists of the equilibrium problem for a deformable, inelastic and dissipative solid with the heat conduction problem appended in addition. The variational formulation allows for general dissipative solids, including finite elastic and plastic deformations, non-Newtonian viscosity, rate sensitivity, arbitrary flow and hardening rules, as well as heat conduction. We show that a joint potential function exists such that both the conservation of energy and the balance of linear momentum equations follow as Euler–Lagrange equations. The identification of the joint potential requires a careful distinction between equilibrium and external temperatures, which are equal at equilibrium. The variational framework predicts the fraction of dissipated energy that is converted to heat. A comparison of this prediction and experimental data suggests that α-titanium and Al2024-T conform to the variational framework.

Keywords: Variational principles; Irreversible processes; Coupled problems; Viscoplasticity; Heat conduction

*Corresponding author. Tel.: +1 626 395 4530; fax: +1 626 304 0175.
E-mail address: ortiz@aero.caltech.edu (M. Ortiz).

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1. Introduction

This paper is concerned with the formulation of variational principles characterizing the solutions of the coupled thermo-mechanical problem for general dissipative solids, here understood as the equilibrium problem of an inelastic deformable solid to which the heat conduction problem is appended in addition. Problems of this nature arise in a variety of important fields of application, including: metal forming, machining, casting and other manufacturing processes; high-velocity impact such as ballistic penetration; and others. By a general dissipative solid we understand a deformable solid, possibly undergoing large deformations, possessing viscosity, internal processes and capable of conducting heat. By a variational characterization of the thermo-mechanical problem we specifically mean the identification of a functional whose stationary points are solutions of the problem. Once this functional is known, the stable solutions of the problem may be identified with certain extrema of the functional, should any exist.

Following the pioneering work of Biot (1956, 1958), the variational form of the coupled thermoelastic and thermoviscoelastic problems has been extensively investigated (Herrmann, 1963; Ben-Amoz, 1965; Oden and Reddy, 1976; Molinari and Ortiz, 1987; Batra, 1989). In addition, at present there are well-developed variational principles for the equilibrium problem of general dissipative solids in the absence of heat conduction (Han et al., 1997a, b; Ortiz and Stainier, 1999). By contrast, the case of thermo-mechanical coupling in dissipative materials has received comparatively less attention (cf., Simo and Miehe, 1992; Armero and Simo, 1992, 1993, for notable exceptions).

When the equilibrium and heat conduction problems for general dissipative solids are combined, the resulting coupled problem lacks an obvious variational structure. This lack of variational structure reveals itself upon linearization of the coupled problem, which results in a non-symmetric operator. This essential difficulty accounts for the lack of variational formulations of the coupled thermo-mechanical problem for general dissipative solids. However, in this paper we show that an *integrating factor* exists which delivers the sought variational structure. The ability to identify such an integrating factor hinges critically on a careful distinction between two types of temperature: an *equilibrium temperature*, which follows as a state variable; and an *external temperature*, which equals the equilibrium temperature at equilibrium. Specifically, we investigate integrating factors that follow from a temperature-dependent rescaling of time in all rate processes. We show that, within this class, the integrating factor is unique. Once the requisite integrating factor is identified, potentials can be determined whose critical points are the solutions of the coupled thermo-mechanical problem in both its rate and incremental forms.

The ability to recast the coupled thermo-mechanical problem in variational form has a number of consequences and some beneficial effects. For instance, the variational framework opens the way for the application of the tools of calculus of variations to the analysis of the solutions of the problem. In particular, conditions for the existence and uniqueness of solutions follow from the direct method of the calculus of variations (cf., e.g., Dal Maso, 1993). In addition, localization phenomena such as shear bands can be effectively studied within the framework of free-discontinuity problems (cf., e.g., Braides, 1998; Ambrosio, 2000). A variational statement of the problem also facilitates the formulation of numerical approximations, e.g., by means of Galerkin or Rayleigh–Ritz methods. In addition, in its time-discretized form the variational framework leads to the