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On divergence-free stress fields and zero-energy stress functions in elasticity

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ABSTRACT

Applying two identities for divergence-free non-symmetric and symmetric second-order tensors, novel type of first- and second-order stress functions are proposed for three-dimensional elasticity problems. It is shown that self-equilibrated but non-symmetric 3D stress fields can be generated by one first-order stress function vector, whereas a self-equilibrated and symmetric 3D stress field can be generated by one Airy-type second-order stress function. Assuming linearly elastic materials, the zero-energy modes of the stress functions introduced are derived and investigated. It is pointed out that the structure of the zero-energy modes of the proposed first-order stress function vector is the same as that of the rigid-body displacements in the linear theory of elasticity.

1. Introduction

Divergence-free second-order tensor fields are often used in solid and fluid mechanics and in many other areas of physics. A self-equilibrated stress field in elasticity, for example, is a divergence-free tensor field and can be generated from stress functions by differentiation. The concepts of first- and second-order stress functions have been introduced by [3–5], see also [7]. If symmetry of the shear stresses is not *a priori* required, a divergence-free second-order stress tensor can be generated by a first-order stress function tensor that has six independent non-zero components [4,9,2]. The stress components in that case can be obtained by the combinations of first-order partial derivatives of the stress functions. A symmetric and self-equilibrated stress fields can be obtained from a second-order stress function tensor that has three independent non-zero components [10,4]. The stress components in that case can be obtained by the combinations of second-order partial derivatives of the stress functions. The main advantage in the use of first-order stress functions, from the point of view of finite element analysis, is that they require only C_0-continuous approximations, in contrast to the C_1-continuity requirement for second-order stress functions.

The importance of the knowledge of the zero-energy stress functions in the solution of boundary value problems using complementary energy-based dual- and dual-mixed variational principles and finite element models is similar to that of the zero-energy displacements in the strain energy-based primal- and primal-mixed formulations and finite elements. In the two-dimensional case, the structure of the first-order stress functions that give zero complementary strain energy is as simple as that of the zero-energy displacements (that give zero strain energy). A complete discussion for the two-dimensional case has been given in [4,6] by pointing out that the zero-energy first-order stress functions in 2D have the same structure as that of the rigid-body displacements, and the zero-energy modes are three in number. The structure of the zero-energy first-order stress function modes in the general three-dimensional case is, however, rather complicated, see [1], and their suppression in stress-based dual-mixed finite element procedures represents an extra difficulty.

The introduction of stress functions that generate divergence-free equilibrated stress fields is usually based on the $\operatorname{div}\operatorname{curl}(.)\equiv 0$ identity, i.e., the divergence of the curl of an arbitrary differentiable tensor field is identically zero. This paper presents and applies two novel identities for constructing three-dimensional divergence-free non-symmetric, as well as symmetric second-order tensor fields. These identities are introduced and proven in Section 2. An application of the identities in elasticity for constructing divergence-free and self-equilibrated stress field is discussed in Section 3. The three-dimensional translational equilibrium equations without symmetry are satisfied by one first-order stress function vector. When symmetry of the stress tensor is additionally required, one Airy-type second-order stress function is introduced

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to generate a self-equilibrated stress field in 3D. The zero-energy modes of the new stress functions are investigated in Sections 3.2 and 3.4. For the special case of 2D elasticity, the proposed stress functions induce the already known relations, as pointed out in Section 3.5.

Notation. Using the summation convention, the position vector of an arbitrary spatial point P is denoted by $\boldsymbol{x} = x_k \boldsymbol{e}_k$, where x_k are the Cartesian coordinates of P and $\boldsymbol{e}_k$ represent an orthonormal basis. The gradient of a smooth vector field $\boldsymbol{v}(\boldsymbol{x}) = v_i(x_k)\,\boldsymbol{e}_i$ is defined as $(\text{grad}\,\boldsymbol{v})_{ij} = v_{i,j}$, $(\text{grad}^T \boldsymbol{v})_{ij} = v_{j,i}$, where a comma followed by the letter i in the subscript indicates partial differentiation with respect to x_i and a T in the superscript stands for the transpose. The divergence of $\boldsymbol{v}(\boldsymbol{x})$ is given by $\text{div}\,\boldsymbol{v} = \text{tr}(\text{grad}\,\boldsymbol{v}) = v_{i,i}$, and the divergence of a smooth second-order tensor field $\boldsymbol{A}(\boldsymbol{x}) = A_{ij}(x_k)\,\boldsymbol{e}_i\,\boldsymbol{e}_j$ is defined as $(\text{div}\boldsymbol{A})_i = A_{ij,j}$. The second-order unit tensor is denoted by $\boldsymbol{1} = \delta_{ij}\,\boldsymbol{e}_i\,\boldsymbol{e}_j$, where δ_{ij} is the Kronecker symbol.

2. Identities for divergence-free second-order tensors

Identity 1. Let $\boldsymbol{v}(\boldsymbol{x})$ be an arbitrary, at least twice continuously differentiable vector field. Let the second-order tensor field $\boldsymbol{A}(\boldsymbol{x})$ be obtained from $\boldsymbol{v}(\boldsymbol{x})$ as

$$\boldsymbol{A}(\boldsymbol{x}) = (\text{div}\,\boldsymbol{v})\,\boldsymbol{1} - \text{grad}^T \boldsymbol{v}, \quad A_{ij}(x_k) = v_{a,a}\,\delta_{ij} - v_{j,i}\,. \tag{2.1}$$

Then $\boldsymbol{A}(\boldsymbol{x})$ is a divergence-free tensor, i.e., $\text{div}\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{0}$.

Proof of Identity 1. Using indicial notation we can write:

$$\begin{aligned}(\text{div}\boldsymbol{A})_i = A_{ij,j} &= \left(v_{a,a}\,\delta_{ij} - v_{j,i}\right)_{,j} \\ &= v_{a,aj}\,\delta_{ij} - v_{j,ij} \\ &= v_{a,ai} - v_{a,ia} \quad \equiv \quad 0.\end{aligned} \tag{2.2}$$

Identity 2. Let $f(\boldsymbol{x})$ be an arbitrary, at least thrice continuously differentiable scalar field. Let the second-order tensor field $\boldsymbol{S}(\boldsymbol{x})$ be obtained from $f(\boldsymbol{x})$ as

$$\boldsymbol{S}(\boldsymbol{x}) = \text{div}\,(\text{grad} f)\,\boldsymbol{1} - \text{grad}\,(\text{grad} f). \tag{2.3}$$

Then $\boldsymbol{S}(\boldsymbol{x})$ is a divergence-free symmetric tensor, i.e., $\text{div}\,\boldsymbol{S}(\boldsymbol{x}) = \boldsymbol{0}$ and $\boldsymbol{S} - \boldsymbol{S}^T = \boldsymbol{0}$.

Proof of Identity 1. The symmetry of $\boldsymbol{S}$ follows from its definition (2.3): using indicial notation,

$$S_{ij}(x_k) = f_{,aa}\,\delta_{ij} - f_{,ij}, \tag{2.4}$$

and this is a symmetric expression with respect to the indices i and j. The divergence of $\boldsymbol{S}(\boldsymbol{x})$ is zero by

$$(\text{div}\,\boldsymbol{S})_{,i} = S_{ij,j} = \left(f_{,aa}\delta_{ij} - f_{,ji}\right)_{,j} = f_{,aai} - f_{,jij} \equiv \quad 0. \tag{2.5}$$

Remark. The divergence-free property of $\boldsymbol{S}(\boldsymbol{x})$ defined by (2.3) also follows from *Identity 1*, if the gradient of $f(x)$ is denoted by $\boldsymbol{v}(x)$.

3. Divergence-free stress fields and zero-energy stress functions

3.1. Equilibrium without symmetry

We consider an elastic body in its deformed configuration denoted by Ω and assume that Ω is a simply-connected domain bounded by a closed surface Γ with outward unit normal $\boldsymbol{n}$. Let the body be subjected to conservative body forces of density $\rho\boldsymbol{b}$ in Ω. Following from the balance of linear momentum [12,10], the three-dimensional equilibrium equation is given by

$$\text{div}\,\boldsymbol{\sigma} + \rho\boldsymbol{b} = \boldsymbol{0}, \quad \boldsymbol{x} \in \Omega, \tag{3.1}$$

where $\boldsymbol{\sigma}(\boldsymbol{x})$ is the Cauchy stress tensor. When the balance of angular momentum, i.e., the symmetry of $\boldsymbol{\sigma}$ is *not a priori* required, the translational equilibrium Eq. (3.1) can, in view of *Identity 1* of Section 2, be satisfied by introducing a first-order stress function vector $\boldsymbol{\chi}(\boldsymbol{x})$:

$$\boldsymbol{\sigma}(\boldsymbol{\chi}) = (\text{div}\,\boldsymbol{\chi})\,\boldsymbol{1} - \text{grad}^T \boldsymbol{\chi} + B\boldsymbol{1}, \tag{3.2}$$

where B denotes the potential of the body forces $\rho\boldsymbol{b}$ defined by

$$\rho\boldsymbol{b} = -\text{grad}\,B, \tag{3.3}$$

and $B\boldsymbol{1}$ is a particular solution to (3.1). Using a Cartesian frame xyz, the components of the equilibrated stress field (3.2) are given by the following expressions:

$$\sigma_x = \frac{\partial \chi_y}{\partial y} + \frac{\partial \chi_z}{\partial z} + B, \tag{3.4}$$

$$\sigma_y = \frac{\partial \chi_z}{\partial z} + \frac{\partial \chi_x}{\partial x} + B, \tag{3.5}$$

$$\sigma_z = \frac{\partial \chi_x}{\partial x} + \frac{\partial \chi_y}{\partial y} + B, \tag{3.6}$$

$$\tau_{xy} = -\frac{\partial \chi_y}{\partial x}, \quad \tau_{yx} = -\frac{\partial \chi_x}{\partial y}\,, \tag{3.7}$$

$$\tau_{yz} = -\frac{\partial \chi_z}{\partial y}, \quad \tau_{zy} = -\frac{\partial \chi_y}{\partial z}\,, \tag{3.8}$$

$$\tau_{xz} = -\frac{\partial \chi_z}{\partial x}, \quad \tau_{zx} = -\frac{\partial \chi_x}{\partial z}\,, \tag{3.9}$$

and the matrix of the equilibrated stress tensor $\boldsymbol{\sigma}$ in the orthonormal basis $\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$ is given by

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \chi_{y,y} + \chi_{z,z} & -\chi_{y,x} & -\chi_{z,x} \\ -\chi_{x,y} & \chi_{z,z} + \chi_{x,x} & -\chi_{z,y} \\ -\chi_{x,z} & -\chi_{y,z} & \chi_{x,x} + \chi_{y,y} \end{bmatrix} + \begin{bmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{bmatrix}. \tag{3.10}$$

The symmetric and the skew-symmetric parts of the stress tensor $\boldsymbol{\sigma}$ in terms of $\boldsymbol{\chi}$ can be written as

$$\text{sym}\,\boldsymbol{\sigma} = (\text{div}\,\boldsymbol{\chi})\,\boldsymbol{1} - \frac{1}{2}(\text{grad}\,\boldsymbol{\chi} + \text{grad}^T \boldsymbol{\chi}), \tag{3.11}$$

$$\text{skw}\,\boldsymbol{\sigma} = \frac{1}{2}(\text{grad}\,\boldsymbol{\chi} - \text{grad}^T \boldsymbol{\chi}) = \text{skw}\,(\text{grad}\,\boldsymbol{\chi})\,. \tag{3.12}$$

These expressions show some similarity to the strain-displacement and rotation-displacement relations of linear elasticity. Note also that $\text{tr}\,\boldsymbol{\sigma}(\boldsymbol{\chi}) = \text{tr}\,(\text{sym}\,\boldsymbol{\sigma}) = 2\,\text{div}\,\boldsymbol{\chi}$ and, in addition, the skew-symmetric parts of $\boldsymbol{\sigma}$ and the gradient of $\boldsymbol{\chi}$ are, according to (3.12), equal.

Remark. No proof exists that any self-equilibrated non-symmetric stress tensor can be expressed by the representation (3.2) using one first-order stress function vector $\boldsymbol{\chi}$, i.e., the completeness of representation (3.2) is an open issue.

3.2. Zero-energy modes of the stress function vector $\boldsymbol{\chi}$

The zero-energy displacements in elasticity are usually called as rigid body modes and their suppression in the solution of boundary value problems, especially when strain energy-based variational principles and numerical methods are applied, is of practical importance. Numerical solution procedures that rely on complementary energy-based dual- and dual-mixed variational principles often require the use of equilibrated stress spaces generated by stress functions. In the latter case the suppression of the zero-energy