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Exact closed-form solutions of some Stefan problems in thermally heterogeneous cylinders

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ABSTRACT

It will be shown that one can obtain from a differential-difference equation reformulation exact closed-form solutions to a class of Stefan problems in cylinders with inverse-square thermal heterogeneity: in particular, new interfacial evolutions were discovered for this class of problems, which Gottlieb's approach [Appl. Math. Lett. 15 (2002) 167–172] had not been able to produce, alongside their associated exact closed-form temperature distributions.

1. Introduction

In a recent paper [1], Gottlieb identified the following exactly solvable Stefan problem which models the temperature distribution $U(r, t)$, up to the interfacial position $R(t)$, in the solid phase of an heterogeneous cylinder with inverse-square thermal heterogeneity undergoing inward solidification:

Problem $\mathcal{P}$: Find $U(r, t)$ and $R(t)$ such that

$$\partial_{rr}U + r^{-1}\partial_r U = \rho\kappa^{-1}r^{-2}\partial_t U \quad \text{in} \quad \Omega(t); \tag{1a}$$

$$\lim_{r\to R(t)^+} \partial_r U = \rho\kappa^{-1}L(R(t))R'(t), \quad R(0) = R_0; \tag{1b}$$

$$U\big|_{r=R_0} = -U_1(t) < 0; \tag{1c}$$

$$\lim_{r\to R(t)^+} U = h(t); \tag{1d}$$

$$U\big|_{t=0} = U_2(r) \quad \text{on} \quad 0 < r < R_0, \tag{1e}$$

where $\Omega(t) = (R(t), R_0] \times [0, \tau]$, τ the time for complete solidification of the cylinder; $c_s x^{-2}$, $c_s \in \mathbb{R}$ the specific heat capacity of the solid heterogeneous material; $\rho = c_s\varrho$, ϱ its density; κ its thermal conductivity; and the vector $\Psi := [L(R(t)), h(t), -U_1(t), U_2(r)]$ of latent heat, boundary and initial data takes the value $[\varpi R^{-2}(t), 0, -1, 0]$, ϖ a real parameter. By a solution to problem $\mathcal{P}$ is meant the pair $(R(t), U(r, t))$ which entirely satisfy Eq. (1). Problem $\mathcal{P}$ is a one-phase reduction of a two-phase problem, one which is approximate when the interfacial temperature is time-dependent. Such an approximate reduction is applicable under melting-point depression and in the limit of the solid-melt conductivity ratio being either very high or far less than unity, see Myers and Font [2].

The central approach taken in that paper consisted of a logarithmic transform which yielded a homogenized version of problem $\mathcal{P}$, set in a slab. Despite the elegance of the approach, it has the obvious limitation of being only applicable to heterogeneous materials with latent heat an inverse proportion to the square of $R(t)$. Collaterally, questions arise as to whether the problem considered by Gottlieb is the only exactly solvable Stefan problem there is for such heterogeneous materials; and if the employed method is adaptable to handling other heterogeneous materials with power-law latent heat.

In this paper, we shall partially address these questions by obtaining exact similarity solutions to problem $\mathcal{P}$ for cases distinct from, and inclusive of, that studied by Gottlieb. Here, the employed technique consists of a differential-difference equation ($\mathcal{D}\Delta\mathcal{E}$) reformulation of Eq. (1a), to which an exact closed-form solution is obtained by Clarkson–Kruskal direct similarity method [3,4]. The lattice reformulation is premised on $U(r, t)$ being the ordinary generating function

$$U(r, t) = \sum_{j=0}^{\infty} \alpha_j(t)\log^j\left(\frac{r}{R(t)}\right) \tag{2}$$

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of $\alpha_j(t)$, reminiscent of the logarithm transform of [1]. This same assumption was deployed in the paper [5] to derive some exact similarity solutions to a class of two-phase Stefan problem in cylinder and spheres, albeit in a heuristic manner. The technique we propose here is formal. It is observed that the differential-difference reformulation of Eqs. (1a) and (1b), courtesy of (2), is

$$R^{-1}(t)R'(t)(j+1)\alpha_{j+1}(t)+(j+1)(j+2)\rho^{-1}\kappa\alpha_{j+2}(t)=\alpha'_j(t);\qquad \alpha_1(t)=\rho\kappa^{-1}L(R(t))R(t)R'(t),\quad R(0)=R_0. \tag{3}$$

The interfacial evolution $R(t)=\exp(-2\beta\sqrt{t})$, $\beta\in\mathbb{R}$ obtained by Gottlieb can be realized if $\alpha_1(t)$ is of a constant proportion to $1/\sqrt{t}$. It will be shown later in this paper that if $\alpha_j(t)$ is set as

$$\alpha_j(t)=t^{-j/2}a_j,\quad a_j\in\mathbb{C}, \tag{4}$$

one can recover the main results of [1] entirely. Consequently, worth entertaining is the question of whether or not Eq. (3) admits more solutions than the pair of $\alpha_j(t)=t^{-j/2}a_j$ and $R(t)=\exp(-2\beta\sqrt{t})$. An answer in the affirmative clearly translates to new solutions to problem $\mathcal{P}$ for materials with inverse square radial thermal heterogeneity, but not necessarily the same latent heat types nor boundary and initial data as in the current literature.

We now outline the contents of the rest of this paper. In Section 2 we furnish the $\mathcal{D}\Delta\mathcal{E}$ reformulation of Stefan problem $\mathcal{P}$, obtain its exact closed-form solutions and give a proof of the asymptotic stability of its solutions. Section 3 presents the exact solutions to problem $\mathcal{P}$ for power-law latent heat under explicit conditions; while Section 4 illustrates some of the results obtained in Section 3.

2. Differential-difference formulations and their solutions

The specific objectives of this section is to give the differential-difference formulation of Problem $\mathcal{P}$ and determine its exact closed-form solutions.

2.1. Differential-difference formulation

The complete differential-difference formulation of Stefan problem $\mathcal{P}$, sieved through the infinite series (2), is the following:

Formulation $\mathcal{F}$: Find $\alpha_j(t)$ and $R(t)$ such that

$$R^{-1}(t)R'(t)(j+1)\alpha_{j+1}(t)+(j+1)(j+2)\rho^{-1}\kappa\alpha_{j+2}(t)=\alpha'_j(t) \tag{5a}$$

$$\alpha_1(t)=\rho\kappa^{-1}L(R(t))R(t)R'(t),\quad R(0)=R_0; \tag{5b}$$

$$\sum_{j=0}^{\infty}\alpha_j(t)\log^j\left(\frac{R_0}{R(t)}\right)=-U_1(t); \tag{5c}$$

$$\alpha_0(t)=h(t); \tag{5d}$$

$$\sum_{j=0}^{\infty}\alpha_j(0)\log^j\left(\frac{r}{R_0}\right)=U_2(r)\quad\text{on}\quad 0<r<R_0. \tag{5e}$$

2.2. Solutions to differential-difference formulation

In our attempt to obtain exactly solvable cases of Problem $\mathcal{P}$, following Shen [3] and Clarkson and Kruskal [4], we shall study differential-difference equation (5a) for its exact solutions by assuming that $\alpha_j(t)$ has a separable form

$$\alpha_j(t)=f_j(t)a_j. \tag{6}$$

This subsequently transforms Eq. (3) into

$$\begin{cases}\rho\kappa^{-1}\psi_j(t)a_{j+1}+(j+1)(j+2)\theta_j(t)a_{j+2}=\rho\kappa^{-1}a_j;\\ \psi_j(t)=\dfrac{R'(t)}{R(t)}\dfrac{f_{j+1}(t)}{f'_j(t)},\quad \theta_j(t)=\dfrac{f_{j+2}(t)}{f'_j(t)}.\end{cases} \tag{7}$$

A reduction of Eq. (7) to a constant coefficient equation, which is one of many means of solving it, can be achieved through the introduction of the compatibility recurrence relation

$$f_j(t)=\left(\frac{R'(t)}{R(t)}\right)^j f_0(t). \tag{8}$$

Clearly, from recurrence (8), $\psi_j(t)$ and $\theta_j(t)$ are equal, and as such only those conditions as would make $\psi_j(t)$ constant need be sought. Elementary calculations show that $\psi_j(t)$ is given by

$$\begin{cases}\psi_j(t)=(jr(t)+w(t))^{-1};\\ w(t)=\dfrac{f'_0(t)}{f_0(t)}\left(\dfrac{R(t)}{R'(t)}\right)^2,\\ r(t)=\left(\dfrac{R''(t)}{R'(t)}\left(\dfrac{R(t)}{R'(t)}\right)^2-\dfrac{R(t)}{R'(t)}\right)f_0(t),\end{cases} \tag{9}$$

and is independent of t only if both $r(t)$ and $w(t)$ are. In the following, some special cases leading to $r(t)$ and $w(t)$ being constant are discussed.

2.2.1. Constant r(t) and $f_0(t)$

Suppose that

$$r(t)=m\neq 0,\qquad \lim_{t\to 0^+}R(t)=R_0,\quad \lim_{t\to 0^+}R'(t)=R_1, \tag{10}$$

such that $R_0, R_1\in\mathbb{R}\cup\{\infty\}$. Then the solution to Eq. (5a) is given by

$$\alpha_j(t)=\left(-\sqrt{\left(\frac{R_0}{R_1}\right)^2-2mt}\right)^{-j}a_j, \tag{11}$$

where the a_js are solutions of

$$\rho(j+1)a_{j+1}+\kappa(j+1)(j+2)a_{j+2}=m\rho j a_j, \tag{12}$$

and the conjugate interfacial evolution is governed by

$$R(t)=R_0\exp\left[\frac{1}{m}\left(-\frac{R_0}{R_1}+\sqrt{\left(\frac{R_0}{R_1}\right)^2-2mt}\right)\right]. \tag{13}$$

Eq. (12) does not have a closed form solution, but the generating function $y(x)$ for its solution sequence $\{a_j\}_0^\infty$ verifies the generating ordinary differential equation

$$\begin{cases}\rho(1-mx)\dfrac{dy(x)}{dx}+\kappa\dfrac{d^2y(x)}{dx^2}=0;\\ y(0)=C_1,\ \dfrac{dy(x)}{dx}|_{x=0}=C_2,\end{cases} \tag{14}$$

that is

$$y(x)=C_1+C_2\sqrt{\frac{\pi\kappa}{2m\rho}}\exp\left(-\frac{\rho}{2m\kappa}\right)\times\left[\mathrm{Erfi}\left(\sqrt{\frac{\rho}{2m\kappa}}\right)+\mathrm{Erfi}\left((mx-1)\sqrt{\frac{\rho}{2m\kappa}}\right)\right]. \tag{15}$$

A special case : $\alpha_j(t)=t^{-j/2}(-\beta)^j a_j$

This dependence of $\alpha_j(t)$ on t is a special case of Eq. (11) which results if $R_1=\infty$, $m=-(2\beta^2)^{-1}$, and (consequently) an interfacial