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# Improved model for the U-tube granular instability: Analytical solution and delayed coupling

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ABSTRACT

The U-tube granular instability is understood as a tendency to increase of the height difference $\Delta h$ between the granular material in both branches of a vertically vibrated U-tube. $\Delta h$ has been reported to increase exponentially with time for a certain range of experimental parameters. A simple model based on a cyclic fluid/solid response of the granular material has been used to explain the dependence of the exponential growth rate $\gamma$ with the maximum dimensionless acceleration $\Gamma$ of vibration, at least within a certain range of applicability. We introduce an analytic solution of the model and use it to discuss the effect of several parameters on $\gamma$. The original model could not predict an abrupt decrease of $\gamma$ observed at large $\Gamma$. In this work we show how considering a $\Gamma$ dependent delayed coupling between the granular material and the vibrating tube allows the model to describe the rapid decrease of $\gamma$ seen at large $\Gamma$ and provides a new perspective to understand this behavior.



## 1. Introduction

Vibrated granular materials show a wide display of interesting phenomena, for example segregation, energy localization, Leidenfrost effect, granular fountains, among others. A particularly interesting set of phenomena is the one showing symmetry breaking and spontaneous upward flow. The interest lies not only in the counter-intuitive behavior of the vibrated granular materials and the theoretical challenge to model them, but also on the applied possibilities regarding bulk solids transport [1–4]. One of these phenomena is the granular U-tube instability (see Fig. 1), understood as a tendency to increase the height difference between the level of grains in both branches of a vertically vibrated U-shaped container, partially filled with fine sand [5–12]. Earlier research has identified the interstitial air as a key factor in the development of the instability.

Some theoretical attempts to model the U-tube granular instability have been previously published. Gutman [13] proposed a mechanism based on the development of an horizontal pressure difference between the bottom of the granular columns at each branch. This pressure difference will be responsible for dragging granular material from the lower branch to the tallest. Rajchenbach [14] proposed a mechanism based on an internal convective current, from regions of large compaction to less compacted regions, he used the term dilatant process. Rajchenbach noted that a combination of this internal current and the compensating effect of surface avalanches could explain the formation of a heap (bunkering) in vibrated beds with rectangular containers. In the case of U-tubes, the lack of surface avalanches will let the convective current drive the majority of the material towards one branch of the tube. Neither Gutman nor Rajchenbach implemented their mechanisms in a mathematical model able to make quantitative predictions. A one dimensional model based on a cyclic fluidization scenario was proposed in references [5,7]. This model was able to explain the dependence of $\gamma$ with $\Gamma$, except for cases with large wall friction. Based on the mechanism proposed by Gutman, but using a simpler model for the behavior of a vibrated bed, Clement et al. [9] proposed a model able to describe the time evolution of $\Delta h$ in a case where water was used as the interstitial fluid.

## 2. Cyclic fluidization model

Retaking the theoretical proposal of references [5,7], the tube is assumed to oscillate along a direction parallel to that of the acceleration of gravity, with a time dependent position $z = a\sin(\omega t)$ measured from a stationary reference frame (see Fig. 2), where $a$ is the amplitude, and $\omega = 2\pi f$ with $f$ the frequency of oscillation. During a single oscillation cycle, it is assumed that the material

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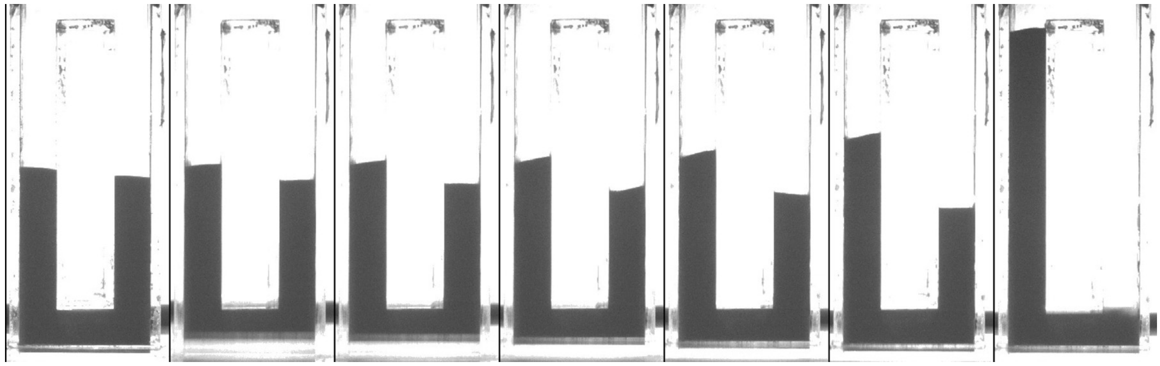

**Fig. 1.** Snapshot sequence showing the evolution of 180-212 μm glass beads in a vertically vibrated U-shaped acrylic container, like the one used in reference [7] (1 s between consecutive images).

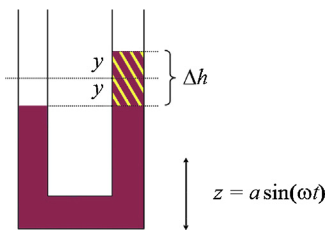


**Fig. 2.** Quantities considered by the cyclic fluidization model.

within the tube behaves as a liquid during the interval $(t_0, t_0+\tau)$, and during the rest of the cycle the material is fully jammed.[1] Here $t_0=\omega^{-1}\arcsin(g/(a\omega^2))$ (with $g$ the acceleration of gravity) is the take off time of a point mass moving on a vibrating plate with position given by $z$ (this is a common assumption for the study of vibrating granular beds). The time during which the granular material is considered as a liquid is $\tau=2\omega^{-1}\arccos(g/(a\omega^2))$, this particular value corresponds to the time during which the effective acceleration felt in a reference frame moving with the container is greater than $g$ in magnitude and points in the opposite direction of gravity. This effective acceleration is $g_{ef}=a\omega^2\sin(\omega t)-g$, therefore $\tau$ is taken as the time during which $g_{ef}>0$.

According to Fig. 2 $\Delta h(t)=2y(t)$. The following one dimensional equation of motion is used to model the evolution of $y$ (see references [5,7] for a more detailed discussion):

$$M_g\frac{d^2y(t)}{dt^2}+\nu\frac{dy(t)}{dt}-2A\rho\left[a\omega^2\sin(\omega t)-g\right]y(t)=0, \tag{1}$$

where $M_g$ is the mass of the granular bed, $\nu$ is a viscous coefficient, $A$ is the cross-sectional area of the tube and $\rho$ is the bulk static density of the granular material. An approximate solution to Eq. (1) was proposed in reference [5] and a full numerical solution was introduced in reference [7]. In the following lines we introduce an analytical solution.

### 2.1. Analytical solution to the equation of motion

Using an integral factor $u(t)=e^{(1/2)\lambda t}$ with $\lambda=\nu/M_g$, we can remove the term $dy(t)/dt$ from Eq. (1) and obtain:

$$\frac{d^2V(t)}{dt^2}+\left[\left(b-\frac{\lambda^2}{4}\right)-c\,\sin(\omega t)\right]V(t)=0, \tag{2}$$

where $V(t)=u(t)y(t)$, $b=2A\rho g/M_g$ and $c=b\Gamma$. After substituting $\omega t=(2z+\pi/2)$, Eq. (2) reduces to the so-called Mathieu equation [2]:

$$\frac{d^2V(z)}{dz^2}+\left[\frac{4b-\lambda^2}{\omega^2}-\frac{4c}{\omega^2}\cos(2z)\right]V(z)=0. \tag{3}$$

Eq. (3) has a general solution given in terms of a linear combination of two special functions known as MathieuC and MathieuS [17]:

$$V(z)=D_1\text{MathieuC}\left(\frac{4b-\lambda^2}{\omega^2},\frac{2c}{\omega^2},z\right)+D_2\text{MathieuS}\left(\frac{4b-\lambda^2}{\omega^2},\frac{2c}{\omega^2},z\right) \tag{4}$$

Returning the change of variables gives the general solution of Eq. (1):

$$y(t)=D_1e^{-\frac{1}{2}\lambda t}\text{MC}(t)+D_2e^{-\frac{1}{2}\lambda t}\text{MS}(t) \tag{5}$$

where the symbols $\text{MS}(t)$ and $\text{MC}(t)$ stand for MathieuS$(((4b-\lambda^2)/\omega^2), (2c/\omega^2), (\omega t/2)-(\pi/4))$ and MathieuC$(((4b-\lambda^2)/\omega^2),(2c/\omega^2), (\omega t/2)-(\pi/4))$ respectively. Constants $D_1$ and $D_2$ follow from imposing initial conditions $y(t_0)=y_0$ and, since the material is supposed to be blocked before the fluidized portion of the cycle, the initial condition for the time derivative of $y$ is $dy(t_0)/dt=0$. Therefore:

$$D_1=y_0e^{\frac{\lambda t_0}{2}}\left[\frac{\text{MS}'(t_0)-(\lambda/\omega)\text{MS}(t_0)}{\text{MS}'(t_0)\text{MC}(t_0)-\text{MC}'(t_0)\text{MS}(t_0)}\right] \tag{6}$$

$$D_2=y_0e^{\frac{\lambda t_0}{2}}\left[\frac{\text{MC}'(t_0)-(\lambda/\omega)\text{MC}(t_0)}{\text{MC}'(t_0)\text{MS}(t_0)-\text{MC}(t_0)\text{MS}'(t_0)}\right] \tag{7}$$

where primed expressions indicate a derivative with respect to $t$.

To illustrate the cyclic fluidization process we show the evolution of the solution $y(t)$ given by expression 5 during four cycles of oscillation at the left side of Fig. 3. Straight horizontal sections correspond to the portions of the oscillation cycle where the granular material is supposed to be blocked and does not flow. Increasing $y$ sections, with a duration of $\tau$ each, correspond to the time during when the solution to the equation of motion is considered. The global initial condition $y_0$ was arbitrarily set to 3 mm. Each subsequent cycle is solved using as new initial condition for $y$ the final position of the previous cycle.

[1] The original suggestion of the cyclic fluidization scheme is due to P. de Gennes, as indicated by Rajchenbach in reference [14], and it has been implemented with success in a model for reverse buoyancy [15]. Experimental evidence has been reported in reference [16].

[2] The Mathieu equation was originally studied to solve the modes of vibration of an elastic membrane with elliptical boundary.