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A note on the classification of anisotropy of bodies defined by implicit constitutive relations

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ABSTRACT

In this note we consider the definition of anisotropy with regard to the response of bodies described by implicit constitutive relations. The class of response relations under considerations in this work is implicit relations between the history of the stress, the history of the density, and the history of the deformation gradient. It is shown that the work of Noll [4] defining the anisotropy of bodies in terms of symmetry groups for Simple Materials can be very easily extended to define the anisotropy in terms of symmetry groups for materials whose response is described by relations between the histories of the stresses and the deformation gradient. While symmetry groups are defined, the more arduous task of developing representation theorems for bodies defined through implicit response relations is an important open task.

1. Introduction

In this short note, we provide the definition of different classes of anisotropies for bodies whose response is given through relations between the histories of the stresses and the deformation gradient. Such a classification is timely as there has been considerable interest in describing the response of bodies using implicit response relations in view of their being able to describe response phenomena that cannot be described within the context of Simple Materials (Noll [4]). Implicit response relations could involve histories of appropriate quantities or just current values for those quantities. For bodies described in general by response relations that relate histories, the notion of what is meant by symmetry and the subsequent development of the appropriate representation have not been put into place. Before one can develop representations, one needs to first understand precisely what is meant by anisotropy within the context of such classes of implicit response relations and it is to this task that this short note is addressed. While this might be a relatively simple task, it provides the starting point for the far more arduous task of determining the representations that are appropriate for the response relations that belong to the different symmetry groups. Even in the case of Simple Materials this task of determining the representations required the efforts of numerous researchers: Rivlin, Green, Spencer, Smith, and many others. In the present case the problem is much more challenging.

Anisotropy of a body is usually interpreted as the response at a point belonging to the body being different, when the body is stimulated along different directions, or put differently the property of the body being different along different directions. The Oxford English Dictionary gives the following definition of anisotropy:

"Anisotropic (æ,naɪsəʊˈtrɒpɪk), a. [mod. f. Gr. ἄνισος unequal + τροπικός belonging to turning, f. τρόπος a turning.]

Acting in different ways to the way of polarized light; possessing the power of right-and left-handed polarization; æolotropic.

1879 Rutley Stud. Rocks ix. 77 Minerals – which exhibit double refraction or are anisotropic. 1881 Maxwell Electr. & Magn. I. 137 A heterogeneous anisotropic medium."

The entries in the Oxford English Dictionary also provide a time line for the usage of a word, and as can be seen from the entry provided above, the Oxford English Dictionary attributes the first usage of the word anisotropic in English, as far as it can tell, to the usage by Rutley and it seems to have been used first to discuss the fact that light rays coming through a polarizer are "turned". The Webster's dictionary defines anisotropy as "exhibiting properties with different values when measured in different directions".

The word "turning" is usually understood as "rotation". The group of rotations comprise the proper orthogonal group, while the set of all rotations and reflections is the full orthogonal group. While in a laboratory one can test a body by rotating it by different amounts, one cannot achieve the reflection of a body and then determine its response to stimuli. However, a body whose structure is an inversion of another body could respond differently to external stimuli, especially stimuli that are not of mechanical

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origin. Moreover, a body whose internal structure is an inversion of another ought to be viewed as a different body. A body and the body that is its inversion are firstly not the same body, secondly they cannot always be envisaged as responding identically when subject to external stimuli.

One could ask several different questions concerning the invariance in the response of a body with regard to its placement. And it is important to be clear concerning which question is being considered as the answers are quite different. The first question is to which sub-class of rotations to which a body is subject to is the body's response invariant? The second question is the more general question, namely to which sub-class of orthogonal transformations is the response invariant? And an even more general question is to which sub-group of unimodular transformations is the response of the body invariant? The first group of transformations, namely the sub-group of rotations can be carried out by an experimentalist and hence can be tested in the laboratory. The second group of transformations that includes inversions of a body that cannot be carried out by an experimentalist and is really a question of mathematical import. While one could carry out a unimodular transformation that is not orthogonal like that of simple shear in a laboratory, invariance to such response is clearly not the intent within which the original ideas of anisotropy were perceived. Thus, in the development of constitutive relations for bodies, one could chose to use any one of these frameworks as long as the context is made apparent. Requiring the invariance of the body to a larger group of transformations leads to a more restrictive class of constitutive relations and hence might inhibit the choice open to the modeler. One can find reasons to support requiring invariance with regard to both the proper orthogonal group as well as the full orthogonal group, but we shall not get into these issues here. We shall address the issue of invariance with respect to sub-groups of rotations in this paper. This is not the invariance considered by a large group of researchers in the field of continuum mechanics. Truesdell [18] and Truesdell and Noll [19] while discussing "material symmetry" introduce the notion of what they refer to as "peer groups" and take the point of view that one ought to discuss the difference or otherwise of the response of materials from a more general standpoint, namely with regard to the body's response on being in reference configurations that may be related to a more general class of transformations than just a sub-group of rotations as seems to be the original intent of the definition. They describe the differences in the response of bodies in two different reference configurations that are related through unimodular transformations, in other words they allow for non-orthogonal transformations between the reference configurations that are unimodular. In fact, Truesdel [18] remarks *"The term "isotropy group", used by NOLL in introducing these groups, is misleading here because it derives from the concept of turning, while the elements of the peer group need not all be rotations; "symmetry", while closer to popular speech of physicists, would be equally misleading because it derives from the concept of distance, which is irrelevant in material response. The term "peer" is intended to suggest its root meaning, which is "equal in status before the law", the "law" being here the constitutive relation of the material"*. It is worth noting that Truesdell is also interpreting "turning" to be rotation, and the notion of isotropy and anisotropy as stemming from rotations. He proposes invariance with respect to sub-groups of a much larger group. While such a generalization with regard to the delineation of invariance with respect to the response might be fine if one is interested in the general invariance of response, it should not be confused with the notion of anisotropy, that is, the response being different along different directions. In this short note, we shall take the more traditional view of discussing the invariance of the response of a body with respect to different directions, that is, we shall restrict ourselves to reference configurations related by rotations. For those that are wedded to the notion that symmetry is determined by sub-groups of the unimodular group, the definitions in this work can be simply extended by substituting the unimodular group for the proper orthogonal group.

The development of implicit constitutive theories to describe the response of materials could be attributed to a generalization of the seminal work of Maxwell [3], though the classical Maxwell model for a viscoelastic fluid with constant material moduli is not a true implicit model as the symmetric part of the velocity gradient can be expressed as a function of the stress and its time derivative. However, if the model were to be generalized wherein the viscosity and the relaxation time are to be functions of the symmetric part of the velocity gradient, that is if the viscosity and the relaxation time depend on the shear rate, the model would be an implicit model. A generalization of the classical Navier-Stokes model to one in which the viscosity is a function of the mean value of the stress and the shear rate would also be an implicit constitutive relation. Fully implicit constitutive relations were developed primarily to describe the response of viscoelastic fluids (see for instance the works of Burgers [1] and Oldroyd [10]) and the inelastic response of solids. With regard to models concerning viscoelastic fluids, as the bodies are isotropic no effort was expended to looking at other types of material symmetry for the body. In the case of inelastic response, a lot of the modeling concerned polycrystalline materials and the elastic response of such bodies was also assumed to be isotropic though anisotropy was considered with regard to the yield surface. In the case of single crystal plasticity however, the anisotropy of the material played an important role in describing the material's response and the anisotropy was captured by introducing directors.

We start by considering a very general class of bodies whose response is given by a functional(an operator) that relates the history of the stresses, the density and the deformation gradient. After obtaining the general invariance requirements, we consider the material symmetry of bodies that are given by implicit functions.

2. Kinematics

We shall now discuss bodies defined through implicit constitutive relations. Mathematicallly speaking, a body is a set that is endowed with a topology and a measure. Such bodies have been defined at various levels of abstraction (see [5–8,18,9]), but all these definitions share one common structure, namely the topological spaces are populated by entities that we refer to as "points", that is the topology that is used is the traditional "point set topology". However, as pointed out recently by Rajagopal [15] there are several problems in natural philosophy wherein the notion of a "point" comes in the way of describing the problem and in resolving them, which could possibly be dealt with within the context of a topology without points. We shall not get into a discussion of these issues here.

Here, we shall merely concern ourselves with the standard definition of a body that is due to Noll and his co-workers, namely one that considers the abstract body to be a set $\mathcal{B}$ endowed with a measure (mass) and the usual topology of open balls in a three dimensional Euclidean space.

By a placer κ, we mean a one to one mapping

$$\kappa : \mathcal{B} \to \mathcal{E}, \tag{1}$$

where $\mathcal{E}$ denotes an Euclidean space of dimension three. $\kappa(\mathcal{B})$ is referred to as a configuration of the body in the three dimensional Euclidean space. By the motion of a body, one refers to a one parameter family of placers κ_t, $t \in \mathcal{R}$, $\mathcal{R}$ being the set of real numbers and the parameter being time. The members $\mathcal{P} \in \mathcal{B}$ are called particles of the abstract body. For a $\mathcal{P} \in \mathcal{B}$ let $\mathbf{X} := \kappa_R(\mathcal{P})$, where κ_R is one of the one parameter family of placers that we shall identify as a reference placement of the body, and $\kappa_R(\mathcal{B})$ being