Mechanics Research Communications 58 (2014) 133–138

Contents lists available at ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

A new variational principle for cohesive fracture and elastoplasticity

Christopher J. Larsen*

WPI, United States

ARTICLE INFO

Article history:
Received 22 June 2013
Received in revised form 26 October 2013
Accepted 30 October 2013
Available online 7 November 2013

Keywords:
Fracture
Plasticity
Variational principle

ABSTRACT

Variational methods for studying cohesive fracture and elastoplasticity have generally relied on minimizing an energy functional that is the sum of a stored elastic energy and a defect energy, corresponding to fracture or plasticity. The usual method for showing existence of minimizers is the Direct Method, whose success requires some properties of the defect energy that are not physically motivated, or in fact are contrary to physically desired properties. Here we introduce a new variational principle based on the idea of "necessity" of the defect, in the spirit of Garroni and Larsen (2009), reflecting the notion that these defects occur only if *necessary* in order for the elastic stress to be admissible, i.e., under the critical stress at which fracture or plasticity begins. The advantage is that the Direct Method only comes into play with a constraint on the defect set, which obviates some of the technical issues usually involved. The most significant advantage is that existence of global minimizers generally requires an infinite stress or strain threshold for plasticity or fracture, while our formulation is appropriate for finite thresholds. A further advantage is that the method produces local minimizers or locally stable states, rather than less physical global minimizers. General existence results will require new methods, but here we easily show existence in one dimension for both static and quasi-static solutions, even when global minima do not exist.

1. Introduction

A common feature of elastoplasticity and cohesive fracture is that deformations are purely elastic until a yield surface is reached, in the form of the boundary of admissible elastic stresses in the case of elastoplasticity, or a critical stress at which cohesive fracture begins. Considering the case of cohesive fracture, the energy of a deformation or displacement u is

$$E(u) := \int_\Omega W(\nabla u)dx + \int_\Gamma \psi([u])d\mathcal{H}^{N-1},$$

where $[u]$ is the jump in displacement, i.e., the size of the crack opening, and Γ is the crack set in the reference configuration Ω. Of course, u can only be a local minimizer of E if $|DW(\nabla u)| \le \psi'(0^+)$ a.e. in Ω. Hence, $c := \psi'(0^+)$ corresponds to a stress threshold for the nucleation of fracture. It then is natural to define equilibria to be local or global minimizers of E, and even consider quasi-static time evolutions based on global minimality (see, e.g., Mielke and Theil, 2004; Dal Maso and Zanini, 2007). However, showing existence for such problems has proven to be very problematic. Indeed, the only rigorous results that have a stress threshold for fracture are based on a regularization. One approach, put forward in Bourdin et al. (2000), is based on the Ambrosio–Tortorelli approximation of the Mumford–Shah energy. Another approach, by Schmidt et al. (2009), is based on "eigenfracture." Both of these approaches are approximations of a sharp-interface model of fracture, and have the feature that the stress threshold for fracture necessarily blows up as the approximation improves. What is missing is a workable sharp-interface model with a finite stress threshold for fracture. Among other things, such a model would allow for the possibility of proving that numerical methods using a stress threshold for fracture converge to some appropriate sharp interface problem. We propose such a model here, introducing a local stability criterion rather than global minimality. In order to motivate it, we first must explain the issues with global minimization.

In order to use the Direct Method of the Calculus of Variations for these so-called "free discontinuity" problems, it is required that every sequence $\{u_n\}$ with bounded energy converge, up to a subsequence, to a function u (in the appropriate space), with $E(u) \le \liminf_{n\to\infty} E(u_n)$, which we will now see necessitates ψ being sub-additive and $\psi'(0) = \infty$, as well as W being quasiconvex with superlinear growth at infinity. One immediate problem is that we can have $\int_\Omega \psi([u])d\mathcal{H}^{N-1} < \liminf_{n\to\infty} \int_\Omega \psi([u_n])d\mathcal{H}^{N-1}$ if ψ is not sub-additive. A more significant problem comes from the spaces involved. The natural space on which to define E is *SBV* introduced in Ambrosio (1989), made up of functions with only smooth and discontinuous variation. If $c < \infty$, then it does not follow from the boundedness of $E(u_n)$ that a limit u is in *SBV* (see Ambrosio et al.,

* Tel.: +1 5088316124; fax: +1 5088315824.
E-mail address: cjlarsen@wpi.edu

0093-6413/$ – see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.mechrescom.2013.10.025

2000). Furthermore, even if $u \in SBV$, the limit of $[u_n]$ in the sense of measures can affect ∇u, destroying lower semicontinuity if $c < \infty$. The underlying cause for both is that the discontinuity sets Γ_n corresponding to u_n can become diffuse in the limit. We will return to this point below.

There is a similar issue in elastoplasticity. In small strain linearized elastoplasticity, the strain (i.e., the symmetrized gradient) is decomposed as

$$Eu = e + p,$$

where e is the elastic strain and p the plastic strain. The stress σ is a function only of the elastic strain, $\sigma = \mathbb{C}e$, where $\mathbb{C}$ is the elasticity tensor. The basic idea of this model is that there is an admissible set K for the stress, and plasticity occurs as the stress begins to leave this set. That is, if no plastic deformation has yet occurred, and u changes incrementally so that $\mathbb{C}(Eu)$ remains in K, then no plastic deformation should occur, i.e., $p = 0$. Note that we can formulate cohesive fracture in a similar way, defining $K := B(0, c)$, the ball centered at zero with radius c, and the constraint is $|\sigma| \in K$.

There is an additional characteristic of plasticity – the model either has hardening, softening, or perfect plasticity. Hardening means that once plastic deformation occurs at a point, the set K grows at that point; softening means the set shrinks, and perfect means it stays the same. The hardening regime necessarily is a model for diffuse plasticity, since (assuming initially K is the same at every point) once plasticity occurs at a point, it will occur in the future at some nearby point where the admissible set is smaller. With softening, once plasticity occurs at a point, it will occur in the future at that same point (that is, further plastic deformation will occur), *instead of* at a nearby point, since K is smaller at the original point.

This elastoplasticity can be modeled (formally) with an energy (see Dal Maso et al., 2006, 2007, 2008a,b)

$$\frac{1}{2}\int_\Omega \mathbb{C}e : e dx + \int_\Omega H(p(x))dx,$$

where H is homogeneous of degree 1 in the case of perfect plasticity, superlinear in the case of hardening, and (morally) sublinear in the case of softening (see, e.g., Dal Maso et al., 2008a,b). In the cases of perfect plasticity and softening (assuming the set K is bounded, or equivalently, the slope of H at zero in each direction is finite), this energy needs to be relaxed if one is seeking its minimization, since this behavior of H at most puts an L^1 bound on minimizing sequences, and so one only has compactness in BV (or BD), as we discussed above. In fact, it is important to note that in the case of softening, this is particularly important, since minimizing this energy naturally leads to concentrations in p on Lebesgue measure zero sets. p in that case is a (vector valued) measure, and for perfect plasticity the last term in the energy is

$$\int_\Omega H\left(\frac{p}{|p|}\right) d|p|,$$

also written in shorthand as $\int_\Omega H(p)$ (only for the 1-homogeneous case). Of course, for softening, this does not work, as $(p/|p|)$ is a unit vector, and so for every H this energy is 1-homogeneous, and so cannot model softening. This is handled in (Dal Maso et al., 2008a,b) by adding a concave softening potential. Our view here is that, as we briefly described above, when there is softening it is natural for the plastic deformation to concentrate on sets as small as possible, which means plastic deformations correspond to jump discontinuities. We can then model plastic deformation using only a concave potential ψ, just as in cohesive fracture, where $\psi'(0)$ gives the critical threshold for plastic nucleation.

In Dal Maso et al. (2006), Dal Maso, DeSimone, and Mora give the first existence result for globally minimizing quasi-static elastoplasticity for perfect plasticity, based on a discrete time minimization procedure, and resulting limit as the time step goes to zero, which is in the general framework of Mielke and Theil (2004), Mielke et al. (2002), see also Ortiz and Repetto (1999). The minimization procedure is inductive: given u_n^i, e_n^i, and p_n^i (where $Eu_n^i = e_n^i dx + p_n^i$), these quantities are found at the next time step t_n^{i+1} by minimizing

$$(e, p) \mapsto \frac{1}{2}\int_\Omega \mathbb{C}e : e dx + \int_\Omega H(p - p_n^i),$$

subject to the new boundary conditions (or loads) at time t_n^{i+1}.

The case of superlinear H has also been successfully studied (see, e.g., Mielke, 2006), as in this case, there is extra compactness and minimizers u will be in a Sobolev space rather than in BD, and p will be a function rather than a measure.

As we already mentioned, the case of softening using global minimization has required relaxation or regularization, such as in Dal Maso et al. (2008b), which adds a viscosity term. Our view here is that the main reason for difficulty in modeling softening is due to global minimality. Put simply, the material "sees" that putting in large plastic deformation will result in a relatively small energy cost, due to softening, even though initially, as the plasticity first began, the cost was not small, and corresponded to what was necessary to keep $\sigma \in K$.

We propose a new variational principle, based on locality, and also in the spirit of the threshold formulation for damage we introduced in Garroni and Larsen (2009). The basic idea is simply that new plastic deformation should occur at a point only if otherwise, the stress would leave the admissible set K (in the isotropic scalar case, this corresponds to $|\nabla u| \le C$ for some $C > 0$ corresponding to K). This naturally leads to the idea of considering the minimal plasticity set (in the strongest sense that is well posed), subject to the condition that given this set, the corresponding (elastic plus plastic) equilibrium, with plastic deformation constrained to be within the designated plasticity set, satisfies the stress admissibility condition.

To further motivate the formulation, we consider a one dimensional quasi-static example, where u is defined on $(0, L)$ with boundary condition $u(0) = 0$ and $u(L) = t$, and $W(\cdot) = (1/2)|\cdot|^2$. Suppose further that the admissible set for elastic strain is $[-1, 1]$. Our formulation would then say that the solution $u(t)$ is purely elastic for all $t \in [0, L]$, i.e., until $|\nabla u| = 1 = \partial K$, *no matter how large L is*, which distinguishes it from global minimization. For $t > L$, our formulation requires there to be plastic deformation somewhere, since otherwise the stress or elastic strain would become inadmissible. We suppose that there is plastic deformation at $x = 1/3$ for $t > L$. We suppose, of course, that u is in elasto-plastic equilibrium at every time, i.e., for $t > L$,

$$\Delta u = 0 \text{ in } (0, 1/3) \cup (1/3, 1) \quad \text{and} \quad \frac{\partial u}{\partial x} = \psi'([u]) \text{ at } (1/3)^{\pm}.$$

This is equivalent to minimizing

$$E(u) := \int_\Omega W(\nabla u)dx + \int_\Omega \psi([u])d\mathcal{H}^{N-1} \tag{1.1}$$

subject to the *constraint* that $[u]$ is nonzero only in $\{1/3\}$.

Now consider the question of whether at some later time, plastic deformation begins at, say, the point $x = 2/3$. Here we propose that in the case of perfect plasticity or softening, the answer should be "no," since if there is no plastic deformation there, the stress threshold will not be crossed, i.e., the admissibility criterion will still be met, since more plastic deformation can occur at $x = 1/3$. So, our point of view is that, as for damage in Garroni and Larsen