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A computational framework for the form-finding and design of tensegrity structures

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ABSTRACT

A new computational framework is proposed for the form-finding and design of tensegrity structures with or without super-stability. The form-finding of tensegrities is formulated as two unconstrained minimisation problems where their objective functions are defined based on eigenvalues of a modified force density matrix. The Nelder–Mead simplex method is then used to solve the minimisation problems. Furthermore, another efficient method is suggested for the interactive form-finding and design of tensegrities with geometrical and force constraints. Examples of the form-finding of tensegrities are presented and the results obtained are compared and contrasted with those analytical results documented in the literature, to verify the accuracy and efficiency of the developed methods.

1. Introduction

Tensegrities are pin-jointed, self-equilibrated frameworks composed of a set of discontinuous compressive elements (struts) floating within a net of continuous tensile elements (cables) with an extensive range of important and novel applications in, for instance, deployable aerospace structures (Tibert and Pellegrino, 2002), architectural and structural design (Motro, 2003; Rhode-Barbarigos et al., 2010), biomechanics (Luo et al., 2008), smart systems (Moored et al., 2011) and advanced engineering materials (Fraternali et al., 2012).

Unlike the regular structures of which the geometries are generally known, tensegrity structures need to be pre-stressed and have a special geometry in order to be stable. The process of determining a suitable pre-stress pattern and its corresponding geometry is called form-finding. Form-finding of tensegrity structures is usually performed through analytical and numerical methods—the numerical methods are more practical for large and irregular tensegrities, while the analytical methods are suitable for tensegrities with a high order of symmetry. Numerical form-finding of tensegrities has been widely studied using different methods. A brief review is given below.

Form-finding of tensegrities has been studied by Motro (1984), employing the dynamic relaxation method. Pellegrino (1986) offered a nonlinear programming approach to the form-finding problem. Masic et al. (2005) developed an algebraic method, based on invariant tensegrity transformations, for the form-finding problem. A finite element approach to the form-finding of tensegrity structures has been presented by Pagitz and Mirats Tur (2009). Estrada et al. (2006), Zhang and Ohsaki (2006) and Tran and Lee (2010, 2013) proposed numerical methods for the form-finding of tensegrity structures that employ either iterative eigenvalues or singular value decompositions of the force density and equilibrium matrices. Koohestani and Guest (2013) developed a new platform for the analytical and numerical form-finding of tensegrities, which effectively uses singular value decompositions of equilibrium and compatibility equations. Form-finding of irregular tensegrities has been presented by Li et al. (2010), using the Monte Carlo method. Rieffel et al. (2009) introduced a special evolutionary form-finding method, Koohestani (2012), Paul et al. (2005) and Xu and Luo (2010) used genetic algorithms and Chen et al. (2012a,b) used ant colony systems for the form-finding of tensegrities. Ehara and Kanno (2010) and Kanno (2011, 2012) used mixed integer programming for the form-finding and optimisation of tensegrity structures under different constraints, including discontinuity of struts, compliance, stress and self-weight loads.

In this paper, an unconstrained optimisation approach is proposed for the form-finding of tensegrity structures. The methods effectively employ spectral decomposition of the force density matrix to directly form tensegrities with or without super-stability. The method requires only connectivity data and a random set of force densities for initialisation, enabling us to form a wide variety of tensegrities with different geometrical and mechanical characteristics. The Nelder–Mead simplex method is used throughout as a

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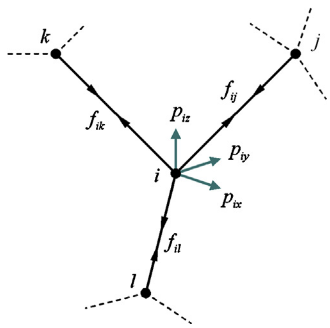

Fig. 1. A typical node of a tensegrity structure.

gradient-free optimiser. Also, an efficient method is introduced for the interactive geometrical and force design of tensegrities. Finally, the viability and efficiency of the methods suggested are studied through some examples of well-known tensegrities.

2. Force density method (FDM)

The FDM, first introduced by Schek (1974), is widely used in the form-finding of tensegrity structures. In this method, equilibrium equations of a tensegrity are written based on the force density of elements and nodal coordinates, leading to a system of linear equations for a known or prescribed set of force densities (for an unknown set of force densities and nodal coordinates, the system is clearly nonlinear). The coefficient matrix of this system of equations, known as the force density matrix is the key object in the study of the stability of tensegrity structures. The force density matrix is also effectively used within our form-finding methods. The formulation is provided briefly, as follows.

In Fig. 1, a typical node (node i) of a 3-dimensional tensegrity is shown. This node is assumed to be connected to the other three nodes (nodes j, k and l) through three elements. Eqs. (1)–(3) provide the static equilibrium equations for node i.

$$\frac{f_{ij}(x_i - x_j)}{L_{ij}} + \frac{f_{ik}(x_i - x_k)}{L_{ik}} + \frac{f_{il}(x_i - x_l)}{L_{il}} = p_{ix} \tag{1}$$

$$\frac{f_{ij}(y_i - y_j)}{L_{ij}} + \frac{f_{ik}(y_i - y_k)}{L_{ik}} + \frac{f_{il}(y_i - y_l)}{L_{il}} = p_{iy} \tag{2}$$

$$\frac{f_{ij}(z_i - z_j)}{L_{ij}} + \frac{f_{ik}(z_i - z_k)}{L_{ik}} + \frac{f_{il}(z_i - z_l)}{L_{il}} = p_{iz} \tag{3}$$

Here, f_{ij} and L_{ij} are force and length of element (i, j), respectively. Furthermore, (x_i, y_j) and z_i are Cartesian coordinates of node i, whilst p_{ix}, p_{iy} and p_{iz} are external forces at node i. Eq. (1) is now simplified using the definition of force density for an element, as follows:

$$(q_{ij} + q_{ik} + q_{il})x_i - q_{ij}x_j - q_{ik}x_k - q_{il}x_l = p_{ix}$$

where $q_{ij} = f_{ij}/L_{ij}$, $q_{ik} = f_{ik}/L_{ik}$ and $q_{il} = f_{il}/L_{il}$ are the force density of elements. Eqs. (2) and (3) can similarly be represented, based on the definitions of the force densities. These equations can be written for all nodes and integrated to form the equilibrium equations of the entire structure, as given in Eq. (4).

$$\mathbf{G}[\mathbf{x}\,\mathbf{y}\,\mathbf{z}] = [\mathbf{p}_x\,\mathbf{p}_y\,\mathbf{p}_z] \tag{4}$$

In Eq. (4), $\mathbf{x} = [x_1, x_2, \ldots, x_n]^t$, $\mathbf{y} = [y_1, y_2, \ldots, y_n]^t$ and $\mathbf{z} = [z_1, z_2, \ldots, z_n]^t$ are vectors of nodal coordinates, whilst $\mathbf{p}_x = [p_{1x}, p_{2x}, \ldots, p_{nx}]^t$, $\mathbf{p}_y = [p_{1y}, p_{2y}, \ldots, p_{ny}]^t$ and $\mathbf{p}_z = [p_{1z}, p_{2z}, \ldots, p_{nz}]^t$ are vectors of external nodal forces in the x-, y- and z-directions, respectively. Moreover, $\mathbf{G} \in \mathbf{R}^{n\times n}$ is known as the force density matrix which, for a tensegrity with n nodes, is a symmetric n-dimensional matrix and can be represented as follows:

$$\mathbf{G} = \mathbf{BQB}^t \tag{5}$$

where $\mathbf{Q} = diag(\mathbf{q})$, $\mathbf{q} = [q_1, q_2, \ldots, q_m]^t$ is the matrix of force densities which, for a tensegrity with m elements, is a diagonal $m \times m$ matrix. Also, $\mathbf{B} = [b_{ij}]_{n\times m}$ is the node-element incidence matrix and is defined as:

$$b_{ij} = \begin{cases} -1 & \text{if } i \text{ is the start node of element } j \\ 1 & \text{if } i \text{ is the end node of element } j \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Note that from the graph-theoretical point of view **G** is, in fact, the Laplacian matrix of a directed weighted graph, where each typical edge (i, j) is directed from node i to $j(i < j)$ and its weight is q_{ij}. Furthermore, the rank of **G**, is at most, $n - 1$ (independent of the numerical values of the force densities), since the sum of all its rows and columns is zero. Eq. (4) describes the equilibrium equations of a 3-dimensional tensegrity structure to which the external forces have been applied. However, in the form-finding, we are interested in the case where the structure has a state of self-stress with no external loads applied; therefore:

$$\mathbf{G}[\mathbf{x}\,\mathbf{y}\,\mathbf{z}] = [0\,0\,0] \tag{7}$$

In general, for a 3-dimensional structure in a state of self-stress, Eq. (7) must have at least three independent solutions. However, because of the intrinsic rank deficiency of **G**, there is always a trivial solution, which is a vector of ones. Thus, the minimum rank deficiency of **G** for a d-dimensional tensegrity is $d + 1$. This also means that **G** must have at least $d + 1$ zero eigenvalues. In the next sections, this feature of the force density matrix is effectively used to formulate our form-finding methods.

3. Stability of tensegrity structures

Tensegrity structures usually exhibit two different forms of stability. The first type of stability, which is called super-stability, is independent of the level of self-stress and type of material in the tensegrity. In general, a d-dimensional tensegrity structure needs to satisfy the following three conditions to be the super-stable (Zhang and Ohsaki, 2007).

(a) The force density matrix **G** has the minimum rank deficiency $d + 1$.
(b) **G** is semi-positive definite.
(c) The rank of the geometry matrix, denoted by **GE**, is $d\,(d + 1)/2$ or, equivalently, the member directions do not lie on the same conic at infinity (Connelly, 1982).

Note that the geometry matrix is defined as

$$\mathbf{GE} = [\mathbf{D}_x\mathbf{d}_x\;\mathbf{D}_y\mathbf{d}_y\;\mathbf{D}_z\mathbf{d}_z\;\mathbf{D}_x\mathbf{d}_y\;\mathbf{D}_x\mathbf{d}_z\;\mathbf{D}_y\mathbf{d}_z] \tag{8}$$

where $\mathbf{d}_x = \mathbf{B}^t\mathbf{x}$, $\mathbf{d}_y = \mathbf{B}^t\mathbf{y}$ and $\mathbf{d}_z = \mathbf{B}_z^t$ are Cartesian components of the length of elements, $\mathbf{D}_x = diag(\mathbf{d}_x)$, $\mathbf{D}_y = diag(\mathbf{d}_y)$ and $\mathbf{D}_z = diag(\mathbf{d}_z)$.

It is useful to note that the first two conditions above play a crucial role in the study of the stability of tensegrity structures, since the third condition is usually satisfied (Zhang and Ohsaki, 2012).

A tensegrity that does not meet above conditions is not super-stable but may still be stable. However, for these cases, the stability can be investigated based on a pre-stress/stiffness ratio of elements and spectral characteristics of the tangent stiffness matrix (sum of the linear elastic stiffness and geometrical stiffness matrices). The reader may refer to Ohsaki and Zhang (2006) for the necessary