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Surface effects in the deformation of an anisotropic elastic material with nano-sized elliptical hole

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ABSTRACT

We examine the effect of surface energy on an anisotropic elastic material weakened by an elliptical hole. A closed-form, full-field solution is derived using the standard Stroh formalism. In particular, explicit expressions for the hoop stress, normal, in-plane tangential and out-of-plane displacement components along the edge of the hole are obtained. These expressions clearly demonstrate the effect of elastic anisotropy of the bulk material on the corresponding field variables. When the material becomes isotropic, the hoop stress agrees with the well-known result in the literature while both the in-plane tangential and out-of-plane displacements vanish and the normal displacement is constant along the entire boundary of the elliptical hole.

1. Introduction

The analysis of an elastically isotropic or anisotropic solid weakened by an elliptical hole is of great practical importance and has been the subject of extensive discussion in the literature (see, for example, Timoshenko and Goodier, 1970; England, 1973; Hwu and Ting, 1989; Wang and Wang, 2006). Traditional modeling attempts (Timoshenko and Goodier, 1970; England, 1973; Hwu and Ting, 1989) have made the assumption that the surface of the elliptical hole is traction-free. Consequently the obtained results are size-independent and cannot explain the experimentally observed size-dependent phenomenon in structures at the nano-scale which exhibit high surface-to-volume ratios. Recently, Wang and Wang (2006) analyzed the problem of an elliptical hole with surface energy in an isotropic elastic solid by using the surface elasticity theory originally proposed by Gurtin and Murdoch (1975) and Gurtin et al. (1998) and by using Muskhelishvili's complex variable formulation (Muskhelishvili, 1953). Wang and Wang (2006) observed that when the size of the hole reduces to the order of the ratio of surface energy to applied stress, the contribution from surface energy becomes significant. Recently, surface elasticity theory has also been incorporated in the study of Eshelby's inclusion problem (Sharma and Ganti, 2004), the inhomogeneity problem (Sharma et al., 2003; Duan et al., 2005; Yang, 2006; Chen et al., 2007), the spherical cavity problem (Yang, 2004) and the crack problem (Kim et al., 2010, Kim et al., 2011; Walton, 2012).

The bulk materials studied in Wang and Wang (2006), Sharma and Ganti (2004), Sharma et al. (2003), Duan et al. (2005); Yang (2006), Chen et al. (2007), Yang (2004), Kim et al. (2010), Kim et al. (2011) and Walton (2012), incorporating surface elasticity were predominantly assumed to be elastically isotropic. In this work, we endeavor to make a rigorous study of an elliptical hole with surface energy in a general anisotropic material by using the Stroh formalism (Ting, 1996). It is interesting to note that the Stroh formalism continues to provide a powerful and elegant method to treat non-standard boundary value problems including the one studied here.

2. Basic formulation

2.1. The Stroh formalism

The equilibrium equations and the stress–strain law for a linear anisotropic elastic material are given by

$$\sigma_{ij,j} = 0, \sigma_{ij} = C_{ijkl}u_{k,l}, \tag{1}$$

where u_i and σ_{ij} are, respectively, the components of displacement and stress while C_{ijkl} are the elastic stiffnesses. Here, we adopt the convention of summation over repeated indices and a comma followed by a subscript, for example, i (i = 1,2,3) denotes the derivative with respect to the ith spatial coordinate;

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For two-dimensional problems in which all quantities depend only on the plane coordinates x_1 and x_2, the general solution can be expressed as (Ting, 1996)

$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T = \mathbf{A}\mathbf{f}(z) + \bar{\mathbf{A}}\overline{\mathbf{f}(z)}, \quad \mathbf{\Phi} = \begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 \end{bmatrix}^T = \mathbf{B}\mathbf{f}(z) + \bar{\mathbf{B}}\overline{\mathbf{f}(z)}, \tag{2}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}, B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}, \quad \mathbf{f}(z) = \begin{bmatrix} f_1(z_1) & f_2(z_2) & f_3(z_3) \end{bmatrix}^T, \quad z_i = x_1 + p_i x_2, Im\left\{p_i\right\} > 0, (i = 1, 2, 3), \tag{3}$$

with

$$\begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix} \begin{bmatrix} \mathbf{a}_i \\ \mathbf{b}_i \end{bmatrix} = p_i \begin{bmatrix} \mathbf{a}_i \\ \mathbf{b}_i \end{bmatrix}, (i = 1, 2, 3) \tag{4}$$

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T, \mathbf{N}_2 = \mathbf{T}^{-1}, \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q}, \tag{5}$$

and

$$Q_{ik} = C_{i1k1}, R_{ik} = C_{i1k2}, T_{ik} = C_{i2k2}. \tag{6}$$

The stress function vector $\mathbf{\Phi}$ is defined, in terms of the stresses, as follows

$$\sigma_{i1} = -\Phi_{i,2}, \sigma_{i2} = \Phi_{i,1}, (i = 1, 2, 3) \tag{7}$$

Due to the fact that the two matrices **A** and **B** satisfy the following normalized orthogonal relationship (Ting, 1996)

$$\begin{bmatrix} \mathbf{B}^T & \mathbf{A}^T \\ \bar{\mathbf{B}}^T & \bar{\mathbf{A}}^T \end{bmatrix} \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} = I, \tag{8}$$

the following three real Barnett–Lothe tensors **S**, **H** and **L** can be introduced (Ting, 1996)

$$\mathbf{S} = \mathrm{i}(2\mathbf{A}\mathbf{B}^T - \mathbf{I}), \mathbf{H} = 2\mathrm{i}\mathbf{A}\mathbf{A}^T, \mathbf{L} = -2\mathrm{i}\mathbf{B}\mathbf{B}^T \tag{9}$$

Furthermore, the two matrices **H** and **L** are symmetric and positive definite, while **SH**, **LS**, $\mathbf{H}^{-1}\mathbf{S}$, $\mathbf{SL}^{-1}$ are anti-symmetric.

In dual coordinate systems (Ting, 1996) where the displacement u_i is referred to the coordinate system x_1, x_2 while the independent variables are referred to a rotated coordinate system: $x_1^* = x_1 \cos\theta + x_2 \sin\theta$ and $x_2^* = -x_1 \sin\theta + x_2 \cos\theta$, we have the following eigenrelation (Ting, 1996):

$$\begin{bmatrix} \mathbf{N}_1(\theta) & \mathbf{N}_2(\theta) \\ \mathbf{N}_3(\theta) & \mathbf{N}_1^T(\theta) \end{bmatrix} \begin{bmatrix} \mathbf{a}_i \\ \mathbf{b}_i \end{bmatrix} = p_i(\theta) \begin{bmatrix} \mathbf{a}_i \\ \mathbf{b}_i \end{bmatrix}, (i = 1, 2, 3), \tag{10}$$

where

$$\mathbf{N}_1(\theta) = -\mathbf{T}^{-1}(\theta)\mathbf{R}^T(\theta), \quad \mathbf{N}_2(\theta) = \mathbf{T}^{-1}(\theta), \mathbf{N}_3(\theta) = \mathbf{R}(\theta)\mathbf{T}^{-1}(\theta)\mathbf{R}^T(\theta) - \mathbf{Q}(\theta), \tag{11}$$

$$p_i(\theta) = \frac{p_i \cos\theta - \sin\theta}{p_i \sin\theta + \cos\theta}. \tag{12}$$

Furthermore $\mathbf{Q}(\theta)$, $\mathbf{R}(\theta)$ and $\mathbf{T}(\theta)$ in Eq. (11) can be expressed as

$$\mathbf{Q}(\theta) = \mathbf{Q}\cos^2\theta + (\mathbf{R} + \mathbf{R}^T)\sin\theta\cos\theta + \mathbf{T}\sin^2\theta, \quad \mathbf{R}(\theta) = \mathbf{R}\cos^2\theta + (\mathbf{T} - \mathbf{Q})\sin\theta\cos\theta - \mathbf{R}^T\sin^2\theta, \quad \mathbf{T}(\theta) = \mathbf{T}\cos^2\theta - (\mathbf{R} + \mathbf{R}^T)\sin\theta\cos\theta + \mathbf{Q}\sin^2\theta. \tag{13}$$

2.2. Surface elasticity

The incorporation of surface elasticity (Wang and Wang, 2006; Yang, 2006, 2004) leads to the following relation between the (2×2) symmetric surface stress tensor $\sigma_{\alpha\beta}^s$ and the deformation-dependent surface energy γ:

$$\sigma_{\alpha\beta}^s = \gamma\delta_{\alpha\beta} + \frac{\partial\gamma}{\partial\varepsilon_{\alpha\beta}}. \tag{14}$$

Here, $\varepsilon_{\alpha\beta}$ is the surface strain tensor and $\delta_{\alpha\beta}$ the Kronecker delta. In this study, we consider only the case in which the surface energy is independent of the elastic strain. This simplification has been adopted by several authors in the literature (see, for example, Wang and Wang, 2006). In doing so, we fully recognize the importance of the strain-dependent contribution to the surface stress in Eq. (14). However, in this paper, we are motivated by an analytical solution of the corresponding problems and our investigations show that such a solution is not available in this more general setting of anisotropic elasticity if we include both the residual (strain-independent) and surface elastic (strain-dependent) contributions to Eq. (14). Nevertheless, as we demonstrate below, interesting phenomena continue to be observed from this simplified model. Consequently, Eq. (14) now reduces to

$$\sigma_{\alpha\beta}^s = \gamma\delta_{\alpha\beta}, \tag{15}$$

which implies that the surface is isotropic (Yang, 2006, 2004).

Using the concept of surface stress, the boundary condition on the surface is given by (Gurtin and Murdoch, 1975; Gurtin et al., 1998; Kim et al., 2011; Ru, 2010)

$$\sigma_{\alpha j} n_j \underline{e}_\alpha + \sigma_{\alpha\beta,\beta}^s \underline{e}_\alpha = 0, \quad \text{(tangential direction)} \qquad \sigma_{ij} n_i n_j = \sigma_{\alpha\beta}^s \kappa_{\alpha\beta}, \quad \text{(normal direction)} \tag{16}$$

where n_i is the unit normal vector of the surface, and $\kappa_{\alpha\beta}$ is the curvature tensor of the surface.

3. An anisotropic material weakened by an elliptical hole in the presence of surface energy

We consider an elliptical hole with surface energy in an infinite anisotropic elastic material. At first we assume that the elastic field is completely induced by surface stress in the absence of external loading. The boundary of the elliptic hole is described by $\Gamma : \left\{\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1\right\}$. In order to solve the boundary value problem, we consider the following mapping functions (Ting, 1996)

$$z_\alpha = x_1 + p_\alpha x_2 = \omega_\alpha(\xi_\alpha) = \frac{a - \mathrm{i}p_\alpha b}{2}\xi_\alpha + \frac{a + \mathrm{i}p_\alpha b}{2\xi_\alpha}, (\alpha = 1, 2, 3) \tag{17}$$

which map the outside of an elliptical region in the z_α-plane onto the outside of the unit circle $\left|\xi_\alpha\right| \geq 1$ in the ξ_α-plane. Meanwhile we consider the mapping function (Muskhelishvili, 1953):

$$z = x_1 + \mathrm{i}x_2 = \omega(\xi) = R\left(\xi + \frac{m}{\xi}\right), \tag{18}$$

where

$$R = \frac{a + b}{2}, \quad m = \frac{a - b}{a + b}. \tag{19}$$

Due to the fact that on the elliptical surface Γ, we have $\xi_1 = \xi_2 = \xi_3 = \xi$, then we can first replace ξ_α by the common variable ξ. After the analysis is complete, the complex variable ξ will revert back to the corresponding complex variables ξ_α, $(\alpha = 1, 2, 3)$.