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On the torsion problem of strain gradient elastic bars

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ABSTRACT

The torsion problem of elastic bars of any cross-sections is discussed, into the context of strain gradient elasticity. It is proven that torsion problem is feasible only for the bars with circular cross-sections. For the other bars (with non-circular cross sections), the non-classical boundary conditions are not satisfied.

1. Introduction

Janmey et al. (1994) has experimentally proved that the shear modulus in the torsion problem depends mainly on the length of the stress fiber. Although that result cannot be explained into the context of mechanics, it has experimentally been verified in the context of conventional elasticity. The author recently, Lazopoulos (submitted for publication) has proved Janmey's conclusion, as far as the variability of shear modulus is concerned, into the context of strain gradient elasticity theory. In fact he proved that the torsion moment does not only depend on the average angle of twist per unit length, but upon the axial length too. Fleck, Hutchinson and their collaborators have applied strain gradient plasticity theories for discussing size effects in various shear (or torsion), fracture and dislocation problems (Fleck and Hutchinson, 1997; Fleck et al., 1994). Let us point out that Toupin (1962) introduced the theories of elasticity with couple stress and Mindlin (1965) restricted the couple stress theory into the context of linear elasticity. However, Aifantis was the first who proposed a simplified strain gradient elasticity theory for studying size effects and lifting of various singularities, especially in fracture and concentrated loading problems, (Aifantis, 1999, 2003) and plasticity as well. Vardoulakis (2004) has introduced strain gradient theories in granular materials (geomaterials). The author has presented various studies concerning beam, plate and shell theories into the context of strain gradient elasticity (Lazopoulos, 2009; Lazopoulos and Lazopoulos, 2010, 2011).

The present work deals with the strain gradient torsion problem of a cylindrical bar of any cross-section. The general problem is discussed with solution satisfying the governing equilibrium equations and the classical and non-classical boundary conditions. It is proved that the non-classical boundary conditions on the cylindrical surface of the bar are not satisfied unless, the cross-section is circular. The torsion of the bars with circular cross-sections have been discussed by the author Lazopoulos (submitted for publication), trying to convince that the deformations of the stress fibers, and generally in cytoskeletal mechanics (Morfat and Kamm, 2006; Lazopoulos and Pirentis, 2007; Lazopoulos and Stamenovic, 2008), should be studied into the context strain gradient elasticity.

2. The strain gradient elasticity

Strain gradient deformation theories have been introduced mainly to describe singular problems in elasticity and plasticity developing boundary layers, and lifting singularities, such as non-smooth displacement fields. An example is the displacement of the point of application of a concentrated load, where conventional elasticity yields infinite displacement fields, whereas the strain gradient elasticity yields infinitesimal ones. Further they have been necessary tools for discussing various problems in micromechanics. The main characteristic is

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the introduction of intrinsic (physical) lengths describing the inhomogeneities or material defects that make those theories adequate to describe problems with evident size effects.

Following a simple version of Mindlin's linear theory of elasticity with microstructure, a widely used micro-elasticity theory, equipped with two additional constitutive coefficients apart from the Lame' constants, is adopted. The intrinsic bulk length g and the directional surface length l_k, see Vardoulakis (2004), are the additional constitutive parameters. Indeed the strain energy density function, for the present case, is expressed by,

$$w = \frac{1}{2}\lambda\varepsilon_{mm}\varepsilon_{nn} + G\varepsilon_{mn}\varepsilon_{nm} + g^2\left(\frac{1}{2}\lambda\varepsilon_{kmm}\varepsilon_{knn} + G\varepsilon_{kmn}\varepsilon_{knm}\right) + l_k\left(\frac{1}{2}\lambda(\varepsilon_{kmm}\varepsilon_{nn} + \varepsilon_{mm}\varepsilon_{knn}) + G(\varepsilon_{kmn}\varepsilon_{nm} + \varepsilon_{mn}\varepsilon_{knm})\right) \tag{1}$$

where, ε_{ij} denotes the infinitesimal strain and ε_{ijk} the infinitesimal strain gradient respectively, with

$$\varepsilon_{ij} = \varepsilon_{ji} = \frac{1}{2}(\partial_i u_j + \partial_j u_i), \quad \varepsilon_{ijk} = \varepsilon_{ikj} = \partial_i \varepsilon_{kj} \tag{2}$$

and $u_i = u_i(x_k)$, the infinitesimal displacement field. In the present work the intrinsic bulk length g will be considered while the lengths l_k will be considered zero.

The constitutive stresses are defined by the relations,

$$\tau_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} \tag{3}$$

and the hyper-stresses by,

$$\mu_{ijk} = \frac{\partial W}{\partial \varepsilon_{ijk}} = g^2(\lambda\varepsilon_{inn}\delta_{jk} + 2G\varepsilon_{ijk}) \tag{4}$$

Further the equilibrium stresses σ_{ij} are defined by,

$$\sigma_{jk} = \tau_{jk} - \frac{\partial \mu_{ijk}}{\partial x_i} = (1 - g^2\nabla^2)(\lambda\varepsilon_{ii}\delta_{jk} + 2G\varepsilon_{jk}) \tag{5}$$

with the equilibrium equations (without body forces)

$$\frac{\partial \sigma_{jk}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\tau_{jk} - \frac{\partial^2 \mu_{ijk}}{\partial x_j \partial x_i}\right) = \partial_j(1 - g^2\nabla^2)(\lambda\varepsilon_{ii}\delta_{jk} + 2G\varepsilon_{jk}) = 0 \tag{6}$$

Further, the boundary conditions (Mindlin, 1965) have been found equal,

$$\begin{aligned} &\overset{1}{\mathbf{t}} = \mathbf{n}(\boldsymbol{\tau} - \nabla\cdot\boldsymbol{\mu}) - \mathbf{L}\cdot[\mathbf{n}\cdot\boldsymbol{\mu}], \quad \text{or} \quad \delta\mathbf{u} = 0 \\ &\overset{2}{\mathbf{t}} = \mathbf{nn}\cdot\boldsymbol{\mu}, \quad \text{or} \quad D\delta\mathbf{u} = 0 \end{aligned} \tag{7}$$

where δ denotes variation, $\mathbf{u}$ is the displacement field and the $\overset{1}{\mathbf{t}}, \overset{2}{\mathbf{t}}$ are the traction and hyper-traction respectively with $\mathbf{n}$ the unit normal vector on the boundary and $D = \mathbf{n}\cdot\nabla$ with $\overset{s}{\nabla} = (\mathbf{I} - \mathbf{nn})\cdot\nabla$ and $\mathbf{L} = \mathbf{n}\overset{s}{\nabla}\cdot\mathbf{n} - \overset{s}{\nabla}$.

The outlined strain gradient elasticity theory will be applied the torsion problem of cylindrical bars.

3. The strain gradient elastic torsion problem of a cylindrical bar

Consider a cylindrical rod with any cross-section and axial length L. Torsion moment M_z is applied at the end cross-sections of the bar. The torsion problem into the context of conventional elasticity is described in all elasticity text, concerning cylindrical bars with any cross-section (Fung, 1965). The strain gradient elastic torsion problem has been studied by Aifantis (1999) and the plastic problem by Fleck and Hutchinson (1997). It has been proven that size effects have been revealed concerning the width of the rod. Nevertheless, the stress fibers (Morfat and Kamm, 2006) exhibit size effects concerning their length. In fact Janmey et al. (1994) has proved experimentally, that the shear modulus G is increased with the length of the stress fiber. That experimental evidence reveals the size effects concerning the axial dimension of the bar too. Fig. 1 shows the torsion problem of a circular bar. The displacement of any point (x, y, z) is defined by,

$$u_1 = -\vartheta(z)y, \quad u_2 = \vartheta(z)x, \quad u_3 = \zeta(z)\varphi(x, y) \tag{8}$$

The strains, in the present case, are defined by:

$$\begin{aligned} &\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{xy} = \varepsilon_{yx} = 0 \\ &\varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\left(-\frac{d\vartheta(z)}{dz}y + \zeta(z)\frac{\partial\varphi(x, y)}{\partial x}\right), \\ &\varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left(\frac{d\vartheta(z)}{dz}x + \zeta(z)\frac{\partial\varphi(x, y)}{\partial y}\right) \\ &\varepsilon_{zz} = \frac{\partial\zeta(z)}{\partial z}\varphi(x, y) \end{aligned} \tag{9}$$