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Mechanics and thermodynamics of surface growth viewed as moving discontinuities

Jean-François Ganghoffer*

LEMTA – ENSEM, 2, Avenue de la Forêt de Haye, BP 160, 54054 Vandoeuvre Cedex, France

A R T I C L E I N F O

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A B S T R A C T

Surface growth is presently described as the motion of a moving interface of vanishing thickness, physically representing the generating cells, separating a zone not yet affected by growth from a domain in which growth has occurred. The jump conditions of density, velocity, momentum, energy, and entropy over the moving front are expressed from the general balance laws of open systems in both physical and material format. The writing of the jump of the internal entropy production in material format allows the identification of a driving force for surface growth, thermodynamically conjugated to the material velocity of the moving front.

1. Introduction

Growth (resp. atrophy) describes the physical processes by which a material of solid body increases (resp. decreases) its size by addition (resp. removal) of mass. A clear distinction is generally made between growth per se, remodeling (change of properties), and morphogenesis (shape changes), a classification suggested by Taber (1995). It is clear that those three aspects of biological evolution are connected to each other. A further clear distinction or rather classification is made between *volumetric growth* referring to processes taking place in the bulk of the material, a situation typical of many physiological or pathological processes, and *surface growth*, describing mechanisms tied to accretion of deposition of mass at a surface. Recent works in the literature lend however, Epstein (2010) such a distinction is not so marked at least from a kinetic point of view, as volumetric and surface growth may simply be two facets of the same reality.

From the biological point of view, *surface growth* refers to mechanisms tied to accretion and deposition of mass occurring mostly in hard tissues, and is an active mechanism in the formation of teeth, seashells, horns, nails, or bones, see Thompson (1992). As pointed out in Skalak et al. (1997) who developed a kinematic viewpoint of surface growth, the growth or atrophy of part of a biological body by the accretion or resorption of biological tissue lying on the surface of the body is a well established fact. Numerous biological tissues develop by surface growth, with situations that can be classified as either growth surface (e.g. nails and horns) or moving growing surface (e.g. seashells, antlers). An extensive review of surface growth models has been exposed in the recent contribution, Ganghoffer (2010).

The point of view advocated in this contribution is surface growth occurring as the motion of an interface within a solid body, across which the mechanical fields experience discontinuities in relation to a flux of nutrients, leading to a production of mass in the zone swept by the interface. Adopting a macroscopic viewpoint, a typical example in mechanobiology is the propagation of a thin transition zone called the growth plate in long bones, see Carter and Beaupré (2001), connecting the metaphyseal bone and the epiphyseal bone (Fig. 1). This plate witnesses a competition between proliferation of chondrocytes and the ossification process (it is found in children and adolescents), and can be considered at first sight as a singular surface moving in a stationary manner. It is clear that this picture is a simplified view, as the real growth plate has a finite thickness and is endowed with a complex microstructure.

Regarding notations, vectors and higher order tensors are denoted using boldface symbols. The physical and material positions are respectively denoted $\mathbf{X}$ and $\mathbf{x}$. The material and physical gradients are written ∇_R and ∇ respectively. The jump and average operators over a surface of discontinuity $S(t)$ are denoted respectively $[a]_S := a^+ - a^-$ and $\bar{a}$:=$1/2(a^+ + a^-)$, for any scalar or tensorial quantity a; the quantities a^+, a^- are the limit of the otherwise continuous quantity a on both sides of $S(t)$. The unit normal to a surface in the Lagrangian (resp. physical) configuration is denoted

* Fax: +33 383595551.
E-mail address: jean-francois.Ganghoffer@ensem.inpl-nancy.fr

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Fig. 1. Radiography of 12 year old child's lower leg showing growth plates at lower ends of tibia and fibula. The growth plate, also coined the chondro-osseous junction, is a hyaline cartilage joining bone to cartilage.

N (resp. **n**). The volume and surface in the actual (resp. referential) configuration are denoted Ω_t, S_t (resp. Ω_R, S_R). The convention of the summation of the repeated index in a monomial is adopted.

2. Kinematics of the growing interface and Hadamard relations

The physical picture underlying the present vision of surface growth is the following: a solid body is swept by an internal interface delimiting a volume of the newly grown material from the volume of the material not yet affected by growth (Fig. 2). The moving front generates new material properties and a new density, resulting from a mass flux across the front, biologically due to the transport of nutrients from the surrounding bulk. The moving interface can be conceived from a biological perspective as a set of generating cells in the vocabulary of Skalak et al. (1997). This situation is representative of internal remodeling, a surface mechanism occurring for instance in bone. The adaptive response of bones to changes in load history is called bone remodeling, Wolff (1892), with a classification as either *internal* or *external remodeling*, Cowin and Van Buskirk (1979). External or surface remodeling results in a change of the external shape of the overall bone structure, and occurs by the resorption or deposition of bone material on the surfaces. To the contrary, internal remodeling refers to the resorption or deposition of bone material only, accompanied by the removal and densification of the architecture of cancellous bone, but not change its overall shape.

The surface growth (generating cells) is moving to the left (Fig. 2) from an initial surface S_0 (at initial time); the position of the surface at time τ (denoted as S_τ) is such that it divides the total volume (at time t) into a subregion already affected by growth (in the direction

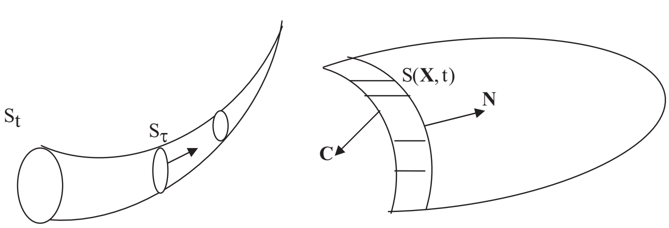

Fig. 2. Surface growth and geometrical domain. The (geometrical) velocity of the moving surface is denoted C. The moving interface can be viewed (right picture) as a region with a vanishing thickness.

of the normal on Fig. 2) and a region not yet affected (in the direction of vector **C**, giving the growth velocity). As the interface is a zone of discontinuity of the fields (density, kinematic and static variables) due to the mass flux it experiences, this is source of a kinematic incompatibility, generating stresses.

As shall be exposed in forthcoming developments, the jump conditions that prevail at the moving surface represent the singular part of the balance laws underlying volumetric growth. One presently imagines that volumetric growth takes place in a narrow zone of a finite thickness (Fig. 2). Adopting a macroscopic viewpoint and in the limit of a very thin such layer, the fields then experience discontinuities at physical positions swept by the front. Volumetric growth may continue to develop in the volume already swept by the interface (or it may stop), but it shall nevertheless not affect what specifically happens locally at the interface. A distinction between the two separate cases of fixed growth surface and moving growth surface has been considered in Skalak et al. (1997); we shall however not distinguish between these two cases in the present contribution, which will be treated under a common umbrella.

The growth surface can in a general situation be conveniently parameterized by convected curvilinear coordinates θ_1, θ_2, which locates a given cell on this surface. The trajectory of any particle on this surface can be generated by a third curvilinear coordinate $\theta_3(t)$, function of time, such that the growth surface has a motion described by an equation of the form

$$\theta_3(t) = f(t) \tag{2.1}$$

A one to one mapping is supposed to exist between the Cartesian coordinates of any point on the moving surface and the curvilinear coordinates,

$$x_{Gi}(t) = x_i(\theta_1, \theta_2, \tau, t) \equiv x_i(\theta_1, \theta_2, \theta_3(t))$$

with $\tau \leq t$ a past instant in the first equality, describing the surface made of the material points generated at a time $t = \tau$. Hence, the material velocity can be written as

$$\dot{X}_{mi} = \left(\frac{\partial x_i}{\partial t}\right)_{\theta_i} \tag{2.2}$$

wherein the curvilinear coordinates are held fixed to follow the same particle.

For a mobile interface separating two media, having a normal velocity $\mathbf{c} = c_N\mathbf{N}$, the normal to the propagating surface is defined from the equation of the surface $S(\mathbf{X},t) = 0$, with $\mathbf{X} = \mathbf{X}(\theta_1,\theta_1,t)$ the coordinate of a material point constrained to lie on the surface. Deriving the equation $S(\mathbf{X},t) = 0$ with respect to time shows the surface equation has a zero convected derivative (derivative of S following the vector field $\dot{\mathbf{X}} = (\partial\mathbf{X}/\partial t)_{\theta_1,\theta_2}$), viz

$$\frac{\delta S}{\delta t} = \frac{\partial S}{\partial \mathbf{X}}.\dot{\mathbf{X}} + \left(\frac{\partial S}{\partial t}\right)_{\mathbf{X}} = 0 \tag{2.3}$$

Since the vector $\partial S/\partial\mathbf{X}$ is orthogonal to the surface S, hence the unit normal vector to S expresses as (this form will be made more specific in the sequel, considering either the material or physical space, with specific notation for the unit normal)

$$\mathbf{N} = \frac{\partial S/\partial \mathbf{X}}{\left\|\partial S/\partial \mathbf{X}\right\|} \tag{2.4}$$

Hence, the normal interface velocity **C** is given in Lagrangian format in terms of its normal component $C_N := \mathbf{C}.\mathbf{N}$, such that

$$C_N = -\frac{1}{\left\|\text{grad}\, S\right\|}\left(\frac{\partial S}{\partial t}\right)_{\mathbf{X}} \tag{2.5}$$