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Higher-order shear beam theories and enriched continuum

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ABSTRACT

The buckling of higher-order shear beam-columns is studied in the light of enriched continuum. We show the equivalence between the enriched kinematics of usual higher-order shear beam theories with the nonlocal and gradient nature of the associated constitutive law. These equivalences are useful for a hierarchical classification of usual beam theories comprising Euler–Bernoulli beam theory, Timoshenko and third-order shear beam theories. A consistent variationnally presentation is derived for all generic theories, leading to meaningful buckling solutions. It is shown that Timoshenko or some other higher-order shear theories can be considered as nonlocal or gradient Euler–Bernoulli theories. The buckling problem of a third-order shear beam-column is analytically studied and treated in the framework of gradient elasticity Timoshenko theory. Some different gradient elasticity Timoshenko models are presented at the end of the paper with available buckling solutions for repetitive structures and microstructured beams.

1. Introduction

Since the beginning of the XXIth century, there has been a considerable interest in the modelling of small-scale structures comprising micro and nanostructures. These small-scale structures present some scale effects that can be captured using nonlocal mechanics (Wang et al., 2010a). Therefore, many recent researches in theoretical mechanics have been focused on the development of theoretical solutions of structural stability (and dynamics) problems, with some additional scale parameters introduced in the nonlocal constitutive law. Some reference nonlocal solutions have been published for beam, plate, shell structural models (see for instance (Wang et al., 2010a)). In a certain sense, one could say that the mechanics community has investigated the small-scale world with specific constitutive laws generally issued of nonlocal mechanics, including gradient or integral-based nonlocal models. On the other hand, it has been also recently shown that some generic structural models used for mechanical or civil engineering problems belong to the class of nonlocal mechanics. Challamel and Wang (2008) used a nonlocal mechanics model to highlight some specific scale effects for a micro and nano-cantilever elastic structural case. This beam model has also been used at the macroscale level for composite beams or sandwich elastic beams (see also (Zhang et al., 2010) or (Challamel and Girhammar, 2011)). Therefore, nonlocality has a wide range of applications from small-scale structures to large-scale structures, even if the source of nonlocality is physically and inherently different. We do not discuss here the nonlocal inelasticity models (plasticity or damage) used for macroscale or microscale structures in presence of softening (see (Bažant and Cedolin, 2003) or more recently (Challamel et al., 2010a) or (Challamel, 2010)). In the present paper, we discuss the buckling of higher-order shear beam-columns in the light of enriched continuum, namely nonlocal and gradient mechanics. We show the equivalence between the enriched kinematics of usual higher-order beam theories with the nonlocal and gradient nature of the associated constitutive law. These equivalences are useful for a hierarchical classification of usual beam theories comprising Euler–Bernoulli beam theory, Timoshenko and third-order shear beam theories.

With respect to integral-based elasticity nonlocal model, Eringen's based nonlocal model (Eringen, 1983) has been shown to be efficient to take into account scale effects at the beam scale for most structural cases (see for instance Peddieson et al. (2003) for the bending problem, or more recently Challamel and Wang (2008). Sudak (2003) obtained the buckling solution for some Euler–Bernoulli beam problems including Eringen's nonlocal terms. These results have been extended to Timoshenko nonlocal columns (see for instance (Wang et al., 2006); (Reddy, 2007)). Wang et al. (2009) investigated the post-buckling problem of cantilevered nano rods/tubes under an end concentrated load. Reddy (2010) gave the general nonlinear formulation of higher-order beam models with Eringen's constitutive law. Challamel and Wang (2010) studied the lateral-torsional buckling problem of Eringen's based

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nonlocal beams. Challamel et al. (2010b) studied the buckling of elastic beams on nonlocal foundation, or the buckling of beam systems with nonlocal elastic connections (Challamel et al., in press). Another family of enriched continuum is the so-called gradient models where the constitutive law also depends on the gradients of the state variables. The in-plane buckling of gradient elasticity Euler–Bernoulli column has been studied by Papargyri-Beskou et al. (2003). Lam et al. (2003) also developed a gradient elasticity Euler–Bernoulli beam and show the stiffening effect of small length terms. Lazopoulos (2003) studied the post-buckling behaviour of gradient elasticity columns. Park and Gao (2006) developed a modified couple stress theory for Euler–Bernoulli kinematics. Kong et al. (2009) developed a general gradient elasticity theory including the modified couple stress theory for the Euler–Bernoulli beam models. Ma et al. (2008) extended the modified couple stress theory of Park and Gao (2006) to Timoshenko beam models. These models are included in the gradient elasticity Timoshenko beam models of Wang et al. (2010b).

2. Euler–Bernoulli beam mechanics

The energy functional for the buckling of "local" Euler–Bernoulli columns is given by:

$$U[w] = \int_0^L \frac{1}{2}EIw''^2 - \frac{1}{2}Pw'^2 dx \tag{1}$$

where L is the length of the column, P is the axial load, w is the deflection, EI is the bending stiffness. The Euler–Bernoulli beam mechanics is based on the one-dimensional constitutive law:

$$M = EI\chi \quad \text{with} \quad \chi = -w'' \tag{2}$$

where M is the bending moment, and χ is the curvature. The stationarity of the energy functional $\delta U = 0$ leads to the differential equations:

$$EIw^{(4)} + Pw'' = 0 \tag{3}$$

with the natural and essential boundary conditions:

$$\left[EIw''\delta w'\right]_0^L = 0 \quad \text{and} \quad \left[-(EIw''' + Pw')\delta w\right]_0^L = 0 \tag{4}$$

For instance, for the pinned-pinned case, the Euler buckling formulae is simply:

$$P = P^E \quad \text{with} \quad P^E = EI\left(\frac{\pi}{L}\right)^2 \tag{5}$$

3. Timoshenko beam mechanics

This part is devoted to the "local" Timoshenko theory that can be understood as a nonlocal Euler–Bernoulli beam theory. In the hierarchical classification, the Timoshenko model can be considered as a superior model that includes the Euler–Bernoulli one when the shear effect can be neglected. In presence of axial loads, Engesser and Haringx's type models have to be distinguished (see Bažant and Cedolin (2003) for instance). A discussion about the hyperelastic formulation of generic Timoshenko models can be found in Reissner (1982), Bažant (2003), Hodges et al. (2006) or Attard and Hunt (2008). The energy functional of Engesser-type Timoshenko column is given by:

$$U[\psi, w] = \int_0^L \frac{1}{2}EI\psi'^2 + \frac{\kappa GA}{2}(w' - \psi)^2 - \frac{1}{2}Pw'^2 dx \tag{6}$$

where ψ is the rotation, G is the shear modulus, A is the total area, κ is the so-called shear coefficient, a dimensionless factor. It should be noted that asymptotic methods may provide a more consistent way to determine the shear stiffnesses without assuming shear correction factors or plane sections remain plane (Yu and Hodges, 2005). The Timoshenko beam mechanics is based on the two-dimensional constitutive law:

$$\begin{pmatrix} M \\ V \end{pmatrix} = \begin{pmatrix} EI & 0 \\ 0 & \kappa GA \end{pmatrix} \begin{pmatrix} \hat{\chi} \\ \gamma \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \hat{\chi} \\ \gamma \end{pmatrix} = \begin{pmatrix} -\psi' \\ w' - \psi \end{pmatrix} \tag{7}$$

where M is the bending moment, V is the shear force, $\hat{\chi}$ is the pseudo-curvature and γ is the shear strain. The stationarity of the energy functional $\delta U = 0$ leads in this case to:

$$\begin{cases} -\kappa GA(w'' - \psi') + Pw'' = 0 \\ EI\psi'' + \kappa GA(w' - \psi) = 0 \end{cases} \tag{8}$$

with the natural and essential boundary conditions:

$$\left[EI\psi'\delta\psi\right]_0^L = 0 \quad \text{and} \quad \left[(-EI\psi'' - Pw')\delta w\right]_0^L = 0 \tag{9}$$

The characteristic length l_c can be introduced as:

$$l_c = \sqrt{\frac{EI}{\kappa GA}} \tag{10}$$

The system of differential equations is now written with such a characteristic length as:

$$\begin{cases} -EI(w'' - \psi') + Pl_c{}^2 w'' = 0 \\ \psi - l_c{}^2\psi'' = w' \end{cases} \tag{11}$$

The second equation shows that the rotation ψ is the nonlocal spatial average variable of the slope angle w':

$$\psi = \overline{w'} \quad \text{with} \quad \overline{w'} - l_c{}^2\overline{w'}'' = w' \tag{12}$$

The Timoshenko theory is clearly a nonlocal Euler–Bernoulli beam theory where the rotation ψ of the cross section is the nonlocal spatial average variable of the slope angle w'. A similar conclusion could be anticipated from the paper of Falsone and Settineri (2011) even if the nonlocal mechanics has not been explicitly used. Furthermore, Eq. (11) leads to the uncoupled differential equation:

$$(EI - Pl_c^2)\psi''' + P\psi' = 0 \quad \text{or} \quad (EI - Pl_c^2)w^{(4)} + Pw'' = 0 \tag{13}$$

leading to the well-known Engesser formulae for most boundary conditions except for instance the fixed-pinned conditions (see Plantema (1966), Ziegler (1982) or Wang et al. (2005)):

$$P^T = \frac{P^E}{1 + P^E/\kappa GA} = \frac{P^E}{1 + (P^E/EI)l_c^2} \tag{14}$$

For instance, for the pinned-pinned case, the Engesser formulae is simplified in:

$$\frac{P^T}{P^E} = \frac{1}{1 + \pi^2(l_c/L)^2} \quad \text{with} \quad P^E = EI\left(\frac{\pi}{L}\right)^2 \tag{15}$$

This is also the formula of the in-plane and out-of-plane buckling problem of Euler–Bernoulli beam model with Eringen's nonlocal law (see for instance Challamel and Wang (2010)). In fact, the nonlocal Euler–Bernoulli constitutive law is written as:

$$M - l_c^2 M'' = EI\chi \quad \text{or equivalently} \quad M = EI\overline{\chi} \quad \text{with} \quad \overline{\chi} - l_c^2\overline{\chi}'' = \chi \tag{16}$$

with χ as the curvature (one can choose $\chi = -w''$). This nonlocal constitutive law can be also presented in an integral format using the Green's operator associated with this differential equation:

$$M(x) = \int_0^L G(x, y)\chi(y)dy \tag{17}$$