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Some simple Cartesian solutions to plane non-homogeneous elasticity problems

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ABSTRACT

The paper presents some simple solutions to plane stress non-homogeneous isotropic elasticity problems described in Cartesian coordinates. The general problem is formulated in terms of the usual Airy stress function allowing spatial variation in Young's modulus while keeping Poisson's ratio constant. The resulting general equation is lengthy and involves various derivatives of the stress function and first and second order derivatives of the modulus distribution. Two inverse schemes are presented which greatly simplify the general governing equation and allow simple exact solutions to be generated. The first scheme employs simple biharmonic polynomial forms for the stress function thereby automatically giving stress fields identical to the homogeneous case. The governing equation is reduced to a form involving only modulus gradation terms and can often be easily solved for the allowable modulus distribution. The resulting displacement fields are then determined by standard methods. A second related method uses special simplifying elastic modulus variation to greatly reduce the general equation thereby allowing simple integration to determine the solution.

1. Introduction

Academic interest in generating solutions to the elasticity field equations for inhomogeneous materials has existed for over half a century. However, more recent and practical attention in this area has occurred due to the fact that materials with continuous spatial variation in mechanical properties can decrease stress concentrations and intensity factors and produce other desirable stress, strain and displacement distributions. This work has led to a new class of engineered materials called *functionally graded materials* (FGM) that are developed with spatially varying properties to suit particular applications (Miyamoto et al., 1999; Suresh, 2001; Birman and Byrd, 2007). Early work on developing elasticity solutions for inhomogeneous problems began to appear in the literature several decades ago. Currently a sizeable collection of papers exists on a variety of solutions to problems of engineering interest including: half-space problems (Gibson, 1967; Booker et al., 1985; Oner, 1990); hollow tube problems (Horgan and Chan, 1999a); rotating disk problem (Horgan and Chan, 1999b); torsion problem (Horgan and Chan, 1999c); beams (Sankar, 2001); antiplane strain problems (Dhaliwal and Singh, 1978; Clements et al., 1997); complex variable methods for plane problems (Wang and Hasebe, 2003); crack problems (Delale and Erdogan, 1983; Ang and Clements, 1987; Parameswaran and Shukla, 1999). Several of these solutions have been re-compiled and summarized in (Sadd, 2009). Closed-form, analytical solutions for graded materials provide important verification problems for computational modeling, and recent papers developing *graded finite element analysis* (Santare and Lambros, 2000; Kim and Paulino, 2002; Buttlar et al., 2006) indicate growing interest in this area. However, many of the existing analytical solutions require moderate to extensive evaluation for comparison use. Thus the purpose of this paper is to not only provide additional inhomogeneous elasticity solutions to the existing collection, but also to add solutions that can be easily used for verification of computational schemes.

In this paper, some simple Cartesian solutions to plane non-homogeneous elasticity problems are presented. The general problem is first formulated in terms of the usual Airy stress function allowing spatial variation in Young's modulus. Since past experience has generally shown that variation in Poisson's ratio leads to only small changes in the stress field, this elastic parameter is kept constant. The resulting general equation is lengthy and involves various derivatives of the stress function and first and second order derivatives of the modulus distribution. We explore two inverse schemes that greatly simplify the general equation and thus allow simple exact solutions to be generated. The first scheme employs simple biharmonic polynomial forms for the Airy stress function and this reduces the governing equation to include terms involving only gradients of the elastic modulus. Specific cases allow simple solutions for the allowable modulus variation. This method thus gives a stress field identical to the homogeneous case but with a modulus gradation that ensures solution to the governing equation for inhomogeneous material. Employing usual integration methods, displacements are determined that differ from the homogeneous case. This method falls under the more

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general theoretical framework given by Fraldi and Cowin (2004). Motivated by the first scheme, a second related method uses special simplifying elastic modulus variation to greatly reduce the general equation thereby allowing simple integration to determine the solution for the stress function and other field variables. After setting up the problem formulation, four specific solution examples are developed and evaluated.

2. Problem formulation

Using Cartesian coordinate formulation in the x, y-plane, the compatibility equation for the plane stress problem of linear elasticity can be written as

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{E}\frac{\partial^2\phi}{\partial x^2}-\frac{\nu}{E}\frac{\partial^2\phi}{\partial y^2}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{1}{E}\frac{\partial^2\phi}{\partial y^2}-\frac{\nu}{E}\frac{\partial^2\phi}{\partial x^2}\right)+2\frac{\partial^2}{\partial x\partial y}\left(\frac{1+\nu}{E}\frac{\partial^2\phi}{\partial x\partial y}\right)=0 \quad (1)$$

where E is the elastic or Young's modulus, ν is Poisson's ratio and ϕ is the usual Airy stress function that is related to the in-plane stresses by

$$\sigma_x=\frac{\partial^2\phi}{\partial y^2},\quad \sigma_y=\frac{\partial^2\phi}{\partial x^2},\quad \tau_{xy}=-\frac{\partial^2\phi}{\partial x\partial y} \quad (2)$$

Exploring non-homogeneous materials with $E=E(x,y)$ and $\nu=$ constant, relation (1) becomes

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{E}\frac{\partial^2\phi}{\partial x^2}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{1}{E}\frac{\partial^2\phi}{\partial y^2}\right)+\frac{\partial^2}{\partial x\partial y}\left(\frac{2}{E}\frac{\partial^2\phi}{\partial x\partial y}\right)-\nu\left[\frac{\partial^2}{\partial x^2}\left(\frac{1}{E}\frac{\partial^2\phi}{\partial y^2}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{1}{E}\frac{\partial^2\phi}{\partial x^2}\right)-\frac{\partial^2}{\partial x\partial y}\left(\frac{2}{E}\frac{\partial^2\phi}{\partial x\partial y}\right)\right]=0 \quad (3)$$

Expanding various derivatives and recombining particular terms allows relation (3) to be expressed as

$$\frac{1}{E}(\nabla^4\phi)+\frac{\partial}{\partial x}\left(\frac{2}{E}\right)\frac{\partial}{\partial x}(\nabla^2\phi)+\frac{\partial}{\partial y}\left(\frac{2}{E}\right)\frac{\partial}{\partial y}(\nabla^2\phi)+\frac{\partial^2}{\partial x^2}\left(\frac{1}{E}\right)\left(\frac{\partial^2\phi}{\partial x^2}-\nu\frac{\partial^2\phi}{\partial y^2}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{1}{E}\right)\left(\frac{\partial^2\phi}{\partial y^2}-\nu\frac{\partial^2\phi}{\partial x^2}\right)+2(1+\nu)\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{E}\right)\frac{\partial^2\phi}{\partial x\partial y}=0 \quad (4)$$

where $\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}$ and $\nabla^4\phi=\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}$. Note that for the homogenous case, relation (4) would reduce to the biharmonic equation $\nabla^4\phi=0$. Now if we assume that the stress function is biharmonic, Eq. (4) reduces to

$$\frac{\partial}{\partial x}\left(\frac{2}{E}\right)\frac{\partial}{\partial x}(\nabla^2\phi)+\frac{\partial}{\partial y}\left(\frac{2}{E}\right)\frac{\partial}{\partial y}(\nabla^2\phi)+\frac{\partial^2}{\partial x^2}\left(\frac{1}{E}\right)\left(\frac{\partial^2\phi}{\partial x^2}-\nu\frac{\partial^2\phi}{\partial y^2}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{1}{E}\right)\left(\frac{\partial^2\phi}{\partial y^2}-\nu\frac{\partial^2\phi}{\partial x^2}\right)+2(1+\nu)\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{E}\right)\frac{\partial^2\phi}{\partial x\partial y}=0 \quad (5)$$

Notice that governing Eqs. (4), (5) contain terms with first and second gradients of the reciprocal of the elastic modulus. We wish to explore problems which will provide simplifications to these governing relations and thereby allow simple solutions to be established. For example, if we choose biharmonic stress functions with sufficient simplicity, relation (5) will reduce to a simple expression containing only derivatives of the elastic modulus that can be integrated. We investigate some simple stress function cases that include polynomial functions of x and y that are commonly used in homogeneous elasticity solutions (Sadd, 2009). An additional example will use a special modulus variation to reduce the general equation thereby allowing simple integration to determine the solution.

3. Example solutions using biharmonic stress functions

We now explore several special cases with particular biharmonic functional forms for the Airy stress function. The elastic modulus gradation is then determined from the compatibility equation.

3.1. Case 1: $E=E(x)$ and $\phi=Ty^2/2$

Note that for this case $\nabla^4\phi=0$ and the stress field is uniaxial $\sigma_x=T$ $\sigma_y=\tau_{xy}=0$. This case would correspond to a uniaxial tension problem as shown in Fig. 1 where the gradation direction is parallel to the loading direction. For convenience we choose the unit square domain $0\leqslant x,y\leqslant 1$

Eq. (5) reduces to

$$\frac{d^2}{dx^2}\left(\frac{1}{E}\right)=0 \quad (6)$$

which implies that $\frac{1}{E}=Ax+B$ or $E=\frac{1}{Ax+B}$, where A and B are arbitrary constants. Thus we find a restriction on the allowable form of the material grading in order to preserve the simplified uniform stress field found in the homogeneous case. It will be more convenient to rewrite relation (6) in the form

$$E=\frac{E_o}{1+Kx} \quad (7)$$

where E_o is the modulus at $x = 0$ and K is another arbitrary constant related to the level of gradation. Note that $K = 0$ corresponds to the homogeneous case with $E=E_o$.

The displacement field associated with this stress distribution is found using standard procedures incorporating Hooke's law and the strain–displacement relations

$$\begin{aligned}\frac{\partial u}{\partial x}&=e_x=\frac{1}{E}(\sigma_x-\nu\sigma_y)=\frac{T}{E}\\ \frac{\partial v}{\partial y}&=e_y=\frac{1}{E}(\sigma_y-\nu\sigma_x)=-\nu\frac{T}{E}\\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}&=2e_{xy}=\frac{\tau_{xy}}{\mu}=0\end{aligned} \quad (8)$$

These results can then be easily integrated to get

$$u=\frac{T}{E_o}\left(x+K\left(\frac{x^2}{2}+\nu\frac{y^2}{2}\right)\right),\quad v=-\nu\frac{T}{E_o}(1+Kx)y \quad (9)$$

where the usual functions of integration have been chosen to have zero rigid body motion at the origin. Note the somewhat surprising result that the horizontal displacement u also depends on y, a result coming from the fact that the shear strain must vanish.

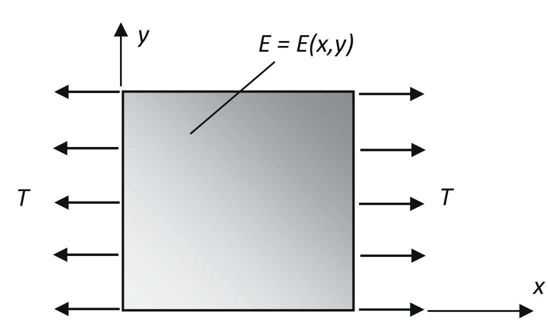

Fig. 1. Uniaxial tension of an inhomogeneous sheet.