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Vibration characteristics of a bi-density drumhead

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Abstract

The *dayan tabla* is a percussion instrument used as an accompaniment to classical Indian singing or music. In this work, a mathematical model was developed to describe the response of this instrument and to analytically determine the areal density distribution required to obtain overtones that were rational multiples of each other. The model was verified by comparing the exact Bessel function solution of the constant density drum with the results obtained using the approximate method outlined in this paper. The maximum error obtained for the first 10 natural frequencies was less than 1% while the associated mode shapes differed by less than 5%. The bi-density model of the tabla, with a mass ratio of 4.1 and a radius ratio of 0.49, predicted natural frequency ratios of approximately 1.1:2:3 (twice):4 (twice) for the fundamental and the first few overtones. This compares well with previously reported experimental data.

Keywords: Tabla; Mathematical model; Galerkin; Modal properties

1. Introduction

Percussion instruments use the natural vibrations of solids or stretched membranes to produce musical sounds. Some percussion instruments provide rhythmic tempo to music (e.g. drum sets, cymbals). Others provide harmony to a musical arrangement (e.g. the steel pan, the xylophone, the tabla) based on its vibration characteristics (normal modes). This latter class of instruments must provide a strong sense of pitch to the musical arrangement. In general, a flexible vibrating body has an infinite number of normal modes. The sound emitted from the excited instrument consists of a frequency spectrum which spans the entire range of these natural frequencies. The lowest frequency in the spectrum is called the fundamental and the others are called overtones or partials. When a note is played, the instrument vibrates with a dominant mode and several higher but less pronounced overtones. If the associated natural frequencies are rational multiples of each other, the sound conveys a strong sense of pitch to the listener and the note is said to be harmonic. For some percussion instruments (such as the tabla), several of the higher notes consist of combinations of the higher normal modes i.e. they do not include the fundamental frequency. For such instruments, it is important that the overtones all be rational multiples of the first partial frequency.

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Tablas are traditional percussion instruments native to North India, usually played in pairs. The larger drum, the bayan, is typically metallic, while the smaller, the dayan, is usually wooden. Each drum is constructed by stretching animal skin over a circular frame, resulting in a tensioned membrane, which is then mass loaded with a paste of charcoal, rice and iron oxide: – the bayan eccentrically and the dayan centrically. The dayan may therefore be considered to consist of a circular membrane attached at its outer edge to an annular membrane of lower density. Previous researchers (Raman (1934), Rossing and Sykes (1982)) have shown experimentally that Indian drums have overtones which are in the approximate ratio 2:3:4:5. This paper determines the optimal radius ratio of the outer annulus and the optimal areal density ratios for the unloaded and loaded sections of the drum, so that the higher overtones are within half a semitone of rational multiples of the first partial. The paper assumes that the tension is constant throughout the membrane and that the mass loading is a function of radius only.

Several researchers (Gutierrez et al., 1998; Laura et al., 1997; Subrahmanyam and Sujith, 2001; Willatzen, 2002) have examined the axisymmetric vibration characteristics of circular membranes with areal densities which are radius dependent; however, only a few (Buchanan and Peddieson, 1999; Buchanan, 2005; Spence and Horgan, 1983) have considered the general motion of such systems. The analysis techniques used include Galerkin's method, integral methods and exact solutions. All these works concentrated on the variation in frequencies as the radius ratio and the areal densities were varied; a theoretical analysis of the mass distribution required to cause the overtones to be harmonic was not determined.

2. Mathematical foundation

The general equation describing the transverse vibration of a fixed outer radius circular membrane with a mass per unit area, $\gamma(\boldsymbol{r})$ and a constant radial tension $\boldsymbol{S}$ is

$$\boldsymbol{S}\left(\frac{\partial^2\hat{\boldsymbol{u}}}{\partial\hat{\boldsymbol{r}}^2}+\frac{1}{\hat{\boldsymbol{r}}}\frac{\partial\hat{\boldsymbol{u}}}{\partial\hat{\boldsymbol{r}}}+\frac{1}{\hat{\boldsymbol{r}}^2}\frac{\partial^2\hat{\boldsymbol{u}}}{\partial\theta^2}\right)=\gamma(\hat{\boldsymbol{r}})\frac{\partial^2\hat{\boldsymbol{u}}}{\partial\hat{\boldsymbol{t}}^2}\quad 0\leqslant\hat{\boldsymbol{r}}\leqslant\boldsymbol{r}_{\max}\tag{1}$$

subject to

$$\hat{\boldsymbol{u}}(\hat{\boldsymbol{r}}_{\max},\theta,\boldsymbol{t})=0$$

where $\hat{\boldsymbol{u}}(\hat{\boldsymbol{r}},\theta,\boldsymbol{t})$ represents the transverse displacement and $(\hat{\boldsymbol{r}},\theta)$ represents the position in polar coordinates. If the system is normalized by setting $\hat{\boldsymbol{u}}=\boldsymbol{r}_{\max}\boldsymbol{u}$; $\hat{\boldsymbol{r}}=\boldsymbol{r}_{\max}\boldsymbol{r}$; $\boldsymbol{c}(\boldsymbol{r})=\sqrt{\frac{S}{\gamma(\hat{\boldsymbol{r}})}}$; $\boldsymbol{t}=\boldsymbol{c}_0\hat{\boldsymbol{t}}$, with $\boldsymbol{c}_0=\boldsymbol{c}(\hat{\boldsymbol{r}}^*)$ being a constant dependent on the tension at a significant radius $\hat{\boldsymbol{r}}^*$, then the equation becomes

$$\boldsymbol{f}(\boldsymbol{r})\frac{\partial^2\boldsymbol{u}}{\partial\boldsymbol{t}^2}=\boldsymbol{L}(\boldsymbol{u})\quad 0\leqslant\boldsymbol{r}\leqslant 1\tag{2}$$

where

$$\boldsymbol{L}(\boldsymbol{u})=\left(\frac{\partial^2\boldsymbol{u}}{\partial r^2}+\frac{1}{\boldsymbol{r}}\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{r}}+\frac{1}{\boldsymbol{r}^2}\frac{\partial^2\boldsymbol{u}}{\partial\theta^2}\right)\tag{2a}$$

and

$$\boldsymbol{f}(\boldsymbol{r})=\left(\frac{\boldsymbol{c}_0}{\boldsymbol{c}(\boldsymbol{r})}\right)^2\tag{2b}$$

The constant density drumhead with fixed outer radius ($\boldsymbol{c}(\boldsymbol{r})=\text{constant}=\boldsymbol{c}_0$) is a classical engineering mathematics/vibrations problem and its solution may be found in many texts (Kreyzig (1999), Meirovitch (2000)). The normal modes of this drumhead, $\boldsymbol{U}_{mn}(\boldsymbol{r},\theta)$, are the products of Bessel's functions of the first kind and sinusoidal functions i.e. $\boldsymbol{U}_{mn}(\boldsymbol{r},\theta)=\boldsymbol{A}_{mn}J_m(\lambda_{mn}\,\boldsymbol{r})\cos(\boldsymbol{m}\theta)$. Here $\boldsymbol{m}$ is the number of nodal diameters and $\boldsymbol{n}$ is the number of nodal circles. However, the ratio between any overtone and the fundamental natural frequency (or the first partial) is irrational. Thus the constant density, constant radial tension drumhead with fixed outer radius cannot be used to produce harmonic notes.