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A comparative analysis of numerical approaches to the mechanics of elastic sheets

Michael Taylor [a], Benny Davidovitch [b], Zhanlong Qiu [b], Katia Bertoldi [a,c,*]

[a] School of Engineering and Applied Sciences, Harvard University, Cambridge, MA, United States
[b] Physics Department, University of Massachusetts Amherst, MA, United States
[c] Kavli Institute for Bionano Science and Technology, Harvard University, Cambridge, MA, United States

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ABSTRACT

Numerically simulating deformations in thin elastic sheets is a challenging problem in computational mechanics due to destabilizing compressive stresses that result in wrinkling. Determining the location, structure, and evolution of wrinkles in these problems has important implications in design and is an area of increasing interest in the fields of physics and engineering. In this work, several numerical approaches previously proposed to model equilibrium deformations in thin elastic sheets are compared. These include standard finite element-based static post-buckling approaches as well as a recently proposed method based on dynamic relaxation, which are applied to the problem of an annular sheet with opposed tractions where wrinkling is a key feature. Numerical solutions are compared to analytic predictions of the ground state, enabling a quantitative evaluation of the predictive power of the various methods. Results indicate that static finite element approaches produce local minima that are highly sensitive to initial imperfections, relying on *a priori* knowledge of the equilibrium wrinkling pattern to generate optimal results. In contrast, dynamic relaxation is much less sensitive to initial imperfections and can generate low-energy solutions for a wide variety of loading conditions without requiring knowledge of the equilibrium solution beforehand.

1. Introduction

Thin elastic sheets are not only found abundantly in nature (Ben Amar and Dervaux, 2008), but are also used in a wide variety of structural applications (Carbonez, 2013) because of their excellent tensional resistance-to-weight ratio. Determining equilibrium deformations in these structures is nontrivial as loading a thin sheet typically results in regions that are locally tense, compressed, or slack (*i.e.*, stress-free). In the compressed regions, wrinkles form in response to instability. Wrinkles may be something engineers wish to avoid (*e.g.*, in solar sails, Vulpetti et al., 2008) or perhaps something that can be used to control membrane behavior (Vandeparre et al., 2010; Breid and Crosby, 2013). In either case, it is of great interest to be able to determine their structure (*i.e.*, amplitude and wavelength) and location in a sheet at a given applied loading state.

Theoretical investigations of deformation and tensional wrinkling in thin sheets go back to the works of Wagner (1929) and Reissner (1938). Early research focused on assuming the sheet to be perfectly flexible and using membrane theory to

* Corresponding author at: School of Engineering and Applied Sciences, Harvard University, Cambridge, MA, United States.
E-mail address: bertoldi@seas.harvard.edu (K. Bertoldi).

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model its deformation (Mansfield, 1968; Stein and Hedgepeth, 1961). These studies formed the foundation of tension-field theory (Pipkin, 1986; Steigmann, 1990; Haseganu and Steigmann, 1994) in which the stress field at the midplane of the sheet is assumed to have no compressive components. Tension-field theory is the appropriate leading-order model at vanishing thickness (Pipkin, 1986; LeDret and Raoult, 1995) and is much more tractable than shell (or higher order) models incorporating bending stiffness (which involve computation of the curvature and its derivatives). However, while tension-field-theory is very useful for assessing the stress distribution and *location* of wrinkled regions in very thin sheets (Haseganu and Steigmann, 1994; Taylor et al., 2014), it offers no information on the actual *structure* of wrinkles. Here, we call the membrane-dominant regime where tension-field theory is valid the "far-from-threshold (FT)" parameter regime. In this regime, the compression induced by the tensile loads is much larger than the thickness-dependent level at which a real sheet buckles. Thus, tension-field-theory cannot describe the *evolution* of a real sheet, upon increasing loads, from a bending-dominant (or "near-threshold" (NT)) regime at the onset of wrinkling to the FT regime, at which a fully wrinkled pattern develops.

In the last decade, several groups have attempted to develop a comprehensive framework that addresses simultaneously the *location, structure* and *evolution* of tensional wrinkle patterns, focusing on a few basic set-ups, such as a rectangular sheet under stretch (Friedl et al., 2000; Nayyar et al., 2011; Puntel et al., 2011; Cerda et al., 2002; Cerda and Mahadevan, 2003; Healey et al., 2013) or shear (Wong and Pellegrino, 2006a,b,c; Zheng, 2009; Diaby et al., 2006), a disk-like (King et al., 2012) or annulus-like sheet (Bella and Kohn, 2014; Géminard et al., 2004; Coman, 2007; Coman and Bassom, 2007; Davidovitch et al., 2011, 2012; Pineirua et al., 2013; Toga et al., 2013) under axially symmetric tensile loads, and a stretched-twisted ribbon (Chopin and Kudrolli, 2013; Chopin et al., 2015). In particular, the first of these examples has attracted considerable interest in the mechanical engineering community (Nayyar et al., 2011). Here, a rectangular-shaped sheet is stretched where its short edges (of width W) are clamped and its long edges (of length L) are free to contract, such that a pattern of parallel wrinkles emerges in a large portion of the sheet, away from the clamped edges. An early numerical work (Friedl et al., 2000) has focused on the onset of wrinkles (*i.e.*, the NT regime) in this system, and results were interpreted by drawing analogy to the classical Euler buckling of rods. Later, Cerda and Mahadevan (2003) addressed the *structure* of this tensional wrinkling pattern away from threshold (*i.e.*, in the FT regime) by drawing an analogy to the elementary example of uniaxially compressing a rectangular sheet on a substrate of stiffness K, which is known to exhibit parallel wrinkles of wavelength $\lambda \approx (B/K)^{1/4}$, where B is the bending modulus of the sheet and K is the substrate's stiffness. The essential observation (Cerda et al., 2002; Cerda and Mahadevan, 2003) was that the presence of tension T along wrinkles of length L induces an effective substrate of stiffness $K = T/L^2$, such that the wavelength of tensional wrinkles satisfies the scaling law: $\lambda \sim (B/T)^{1/4}L^{1/2}$.

Subsequent works by several groups, which addressed the stretched rectangular sheet, have attempted to describe the complete evolution of the wrinkle pattern, as the tensile load is gradually increased from its threshold value to the FT behavior addressed in Cerda and Mahadevan (2003). These studies, however, were encountered by significant difficulties: experimental efforts to probe the onset of the wrinkling instability (Zheng, 2009) were baffled by the high sensitivity of this system to the non-uniformity of the applied loads and the likelihood of plastic deformations on various scales; on the theoretical front the difficulty may be attributed to the lack of analytic solutions of the stress field neither for the planar state (necessary to describe the onset of instability and the NT regime) nor for tension field theory (which provides the basis for analysis of the FT regime). This situation highlights the important role of numerical simulations, even in such a basic system, as the ultimate route for a systematic study of tensional wrinkling. The computational challenge here stems from the multi-scale nature of wrinkling phenomena, whereby the wavelength λ vanishes with the sheet's thickness, while the size of the wrinkled region is determined by the length L of the sheet.

Recognizing the need in reliable, efficient numerical simulations, the primary purpose of this paper is to examine and quantitatively compare the performance of some popular numerical methods for studying the key aspects of tensional wrinkling patterns – their *location, structure*, and *evolution* as the tensile loads are being varied. This purpose dictates our choice of case study, which is known as the Lamé problem (Timoshenko and Goodier, 1970): an annular sheet under radial tensile loads T_{in} and T_{out}, exerted, respectively, on its inner and outer boundaries (see Fig. 1). The key advantage of the Lamé set-up, in comparison to a stretched rectangular sheet, is the existence of analytic predictions for the location and structure of the wrinkle pattern in both NT and FT regimes, as well as the evolution of the pattern between these parameter regimes as the tensile loads are gradually increased. A solution to the stress distribution of the planar state, which can be found in classical textbooks on elasticity theory (Timoshenko and Goodier, 1970), has been attributed to Lamé and has been recently used for linear stability analysis that yields the threshold value of the tensile loads (Coman and Bassom, 2007), as well as the location and structure of the wrinkle pattern in the NT regime. More recently, an exact solution of the tension-field theory equations has been obtained (Coman, 2007; Davidovitch et al., 2011), allowing one to identify exactly the location of the wrinkled zone in a very thin sheet, away from threshold.

The structure of the wrinkle pattern in the corresponding FT regime was described through a singular expansion of the Föppl–von Kármán (FvK) equations around the tension-field-theory stress field, and the evolution from the NT and FT regimes was characterized (Davidovitch et al., 2012). This progress provides us with nontrivial analytic results on the location, structure, and evolution of the wrinkle pattern upon varying the tensile loads. Our hypothesis is that these results provide our best current understanding of the energy-minimizing state for the Lamé problem. Given that the elastic energy has many local minima, the analytical results give a basis for comparing those local minima obtained via simulation methods.