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Stability of pear-shaped configurations bifurcated from a pressurized spherical balloon

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ABSTRACT

It is well-known that for most spherical rubber balloons the pressure versus volume curve associated with uniform inflation is *N*-shaped (the pressure increases rapidly to a maximum, falls to a minimum, and subsequently increases monotonically), and that somewhere along the descending branch of this curve the spherical shape may bifurcate into a pear shape through localized thinning near one of the poles. The bifurcation is associated with the (uniform) surface tension reaching a maximum. It is previously known that whenever a pear-shaped configuration becomes possible, it has lower energy than the co-existing spherical configuration, but the stability of the pear-shaped configuration itself is unknown. With the use of the energy stability criterion, it is shown in this paper that the pear-shaped configuration is unstable under pressure control, but stable under mass control. Our calculations are carried out using the Ogden material model as an example, but it is expected that the qualitative stability results should also be valid for other material models that predict a similar *N*-shaped behavior for uniform inflation.

1. Introduction

Inflating a membrane balloon is a classical problem in Finite Elasticity (Adkins and Rivlin, 1952), which has been studied from a variety of perspectives; see, e.g. Crisp and Hart-Smith (1971), Sagiv (1990), De Tommasi et al. (2013), and the references therein. For most spherical rubber balloons, the pressure versus volume curve associated with uniform inflation has an *N*-shape, and it is well-known that somewhere along the descending branch of this curve (see Fig. 1) the spherical shape may bifurcate into a pear shape through localized thinning near one of the poles; see Feodosev (1968), Needleman (1977), Haughton and Ogden (1978), Haughton (1980), and Chen and Healey (1991). It was shown by Chen and Healey (1991) that whenever the pear-shaped configuration is possible, it has lower energy than the co-existing spherical configuration. This implies, via the energy stability criterion, that the co-existing spherical configuration is unstable, but it does not give any information about the stability of the pear-shaped configuration. Over the remaining section of the descending curve where the pear-shaped configuration is not possible, stability of the spherical configuration has only been studied with respect to spherical perturbations, and it is thought that the spherical configuration is unstable under pressure control but stable under mass control (Alexander, 1971). There also exist a number of studies on the inflation and stability of

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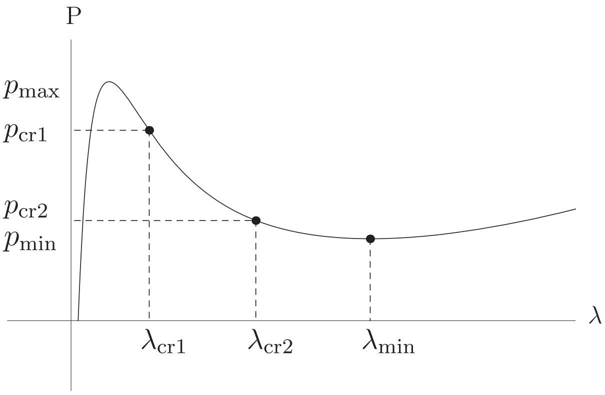

Fig. 1. A typical profile of pressure P versus principal stretch λ in uniform inflation of a spherical shell. Internal volume is proportional to λ^3.

multi-lobed spherical balloons (see, e.g. Crisp and Hart-Smith, 1971; Müller and Struchtrup, 2002), but again, stability is only considered with respect to spherical perturbations.

In this paper, we fill a gap in the literature by studying the stability of both spherical and pear-shaped solutions on the above-mentioned descending branch with respect to *axi-symmetric* perturbations. In addition to its intrinsic theoretical value, the present study may have some relevance to the mathematical modeling of the initiation and final rupture of saccular aneurysms in human arteries (Austin et al., 1989). We also observe that pressurized balloons are increasingly used in a variety of engineering situations, ranging from extra-terrestrial use as foldable habitats (Jenkins, 2001) to microelectromechanical systems as actuators (Youda and Konishi, 2002; Keplinger et al., 2012; Rudykha et al., 2012). Such applications invariably require a good understanding of their stability properties.

Since the inflation problem under consideration is conservative, the main method that we use to assess stability is the energy criterion although its connection with the spectral method is also discussed. More precisely, we calculate the minimum of the second variation of the total energy with respect to all kinematically admissible axi-symmetric perturbations. An inflated configuration is said to be stable if this minimum is positive, and unstable if it is negative. It is generally recognized that the energy stability criterion should be used with caution. For conservative systems, it is known that under appropriate assumptions, the solution being an energy minimizer implies (nonlinear) stability in the Liapunov sense (van der Heijden, 2009), but the converse has not been rigorously proved. In contrast, the spectral method is a very effective method for predicting linear instability. We shall show in this paper that if the above-mentioned minimum is negative, then linear (spectral) instability is implied. For a more rigorous discussion of stability criteria for nonlinear elasticity, we refer to the book by Marsden and Hughes (1993) and the more recent article by Knops (2001).

The rest of this paper is divided into four sections as follows. After formulating the inflation problem, we explain in Section 2 a procedure that can be used to find all axi-symmetric solutions (spherical or pear-shaped in particular) whenever they exist. We then examine in Section 3 the stability, under pressure control, of both the spherical and pear-shaped solutions using a combination of the spectral method and the energy criterion. In Section 4, stability of the pear-shaped solution under mass control is studied. The paper is concluded in the final section with further discussions.

2. Pear-shaped configurations bifurcated from spherical configurations

We consider a spherical balloon that is described by

$$R(\theta)=\sin\theta,\quad Z(\theta)=1-\cos\theta,\ 0\le\theta\le\pi,$$

in terms of cylindrical polar coordinates (R,θ,Z) in its undeformed configuration. Without loss of generality, we have assumed the constant radius to be unity, which is equivalent to using the radius as the unit for length.

We focus on axisymmetric deformations described by

$$r=r(\theta),\quad z=z(\theta), \tag{2.1}$$

where (r,θ,z) are cylindrical polar coordinates in the deformed configuration. This form includes uniformly inflated solutions and pear-shaped bifurcated solutions. Denote by dS and ds the arclengths measured from $\theta=0$ in the undeformed and deformed configurations, respectively. We then have $dS=d\theta$ and $ds=\sqrt{r'^2+z'^2}\,d\theta$, where a prime denotes differentiation with respect to θ. The principal directions of stretch coincide with the lines of latitude, the meridian and the normal to the deformed surface. Thus, the principal stretches are given by

$$\lambda_1=\frac{r}{R},\quad \lambda_2=\frac{ds}{dS}=\sqrt{r'^2+z'^2},\quad \lambda_3=\frac{h}{H}, \tag{2.2}$$

where H and h are the undeformed and deformed thicknesses, respectively.